

# Moving Constraints as Stabilizing Controls in Classical Mechanics

Alberto Bressan<sup>(\*)</sup> and Franco Rampazzo<sup>(\*\*)</sup>

(\*) Department of Mathematics, Penn State University,  
University Park, Pa. 16802, USA.

(\*\*) Dipartimento di Matematica Pura ed Applicata, Università di Padova,  
Padova 35141, Italy.

E-mails: bressan@math.psu.edu and rampazzo@math.unipd.it

April 3, 2008

## Abstract

The paper analyzes a Lagrangian system which is controlled by directly assigning some of the coordinates as functions of time, by means of frictionless constraints. In a natural system of coordinates, the equations of motions contain terms which are linear or quadratic w.r.t. time derivatives of the control functions. After reviewing the basic equations, we explain the significance of the quadratic terms, related to geodesics orthogonal to a given foliation. We then study the problem of stabilization of the system to a given point, by means of oscillating controls. This problem is first reduced to the weak stability for a related convex-valued differential inclusion, then studied by Lyapunov functions methods. In the last sections, we illustrate the results by means of various mechanical examples.

## 1 Introduction

A mechanical system can be controlled in two fundamentally different ways. In a commonly adopted framework [?, ?], the controller modifies the time evolution of the system by applying additional forces. This leads to a control problem in standard form, where the time derivatives of the state variables depend continuously on the control function.

In other situations, also physically realistic, the controller acts on the system by directly assigning the values of some of the coordinates, by means of time dependent constraints. The evolution of the remaining coordinates can then be determined by solving an “impulsive” control system, where the derivatives of the state variables depend (linearly or quadratically) on the time derivative of the control function. This alternative point of view was introduced, independently, in [?] and in [?].

Motivated by this second approach, in the present paper we study the following problem of Classical Mechanics:

*Consider a system where the state space is a product  $\mathcal{Q} \times \mathcal{U}$  of finite-dimensional manifolds  $\mathcal{Q}$  and  $\mathcal{U}$ . Assume that one can prescribe the motion  $t \mapsto \mathbf{u}(t) \in \mathcal{U}$  of the second component,*

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## 1 Introduction

A mechanical system can be controlled in two fundamentally different ways. In a commonly adopted framework [11, 21], the controller modifies the time evolution of the system by applying additional forces. This leads to a control problem in standard form, where the time derivatives of the state variables depend continuously on the control function.

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Motivated by this second approach, in the present paper we study the following problem of Classical Mechanics:

*Consider a system where the state space is a product  $\mathcal{Q} \times \mathcal{U}$  of finite-dimensional manifolds  $\mathcal{Q}$  and  $\mathcal{U}$ . Assume that one can prescribe the motion  $t \mapsto \mathbf{u}(t) \in \mathcal{U}$  of the second component,*

by means of frictionless constraints. Given a point  $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ , can one stabilize the system at this point, by an oscillatory motion of the control  $\mathbf{u}(\cdot)$  around  $\bar{\mathbf{u}}$ ?

A well known example where stability is obtained by vibration is provided by a pendulum whose suspension point can oscillate on a vertical guide, as in Figure 1, left. Calling  $\theta$  the angle and  $h$  the height of the pivot, in this case we have  $(\mathbf{q}, \mathbf{u}) = (\theta, h) \in S^1 \times I$ . Here  $S^1 = [0, 2\pi]$  with endpoints identified, and  $I$  is an open interval. If we take  $\bar{\mathbf{q}} = \bar{\theta} = 0$  as the (unstable) upper vertical position of the pendulum, it is well-known (see for example [1, 14, 15] and references therein) that this configuration can be made stable by rapidly oscillating the pivot around a given value  $\bar{\mathbf{u}} = \bar{h}$ . More generally, we will show that this system can be asymptotically stabilized at any angle  $\bar{\theta}$  with  $-\pi/2 < \bar{\theta} < \pi/2$ , by a suitable choice of the control function  $t \mapsto h(t) = u(t)$ .

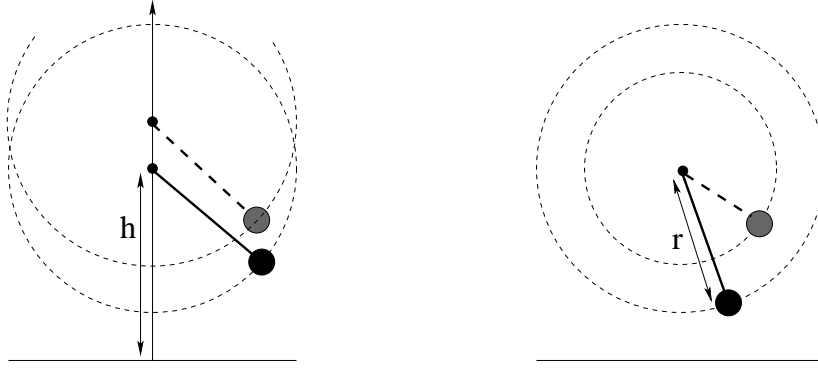


Figure 1: Left: a pendulum with vertically moving pivot and fixed length. Right: a pendulum with fixed pivot and variable length.

On the other hand, consider the variable length pendulum, where the pivot is fixed at the origin, but we can assign the radius of oscillation  $r$  as function of time, see Figure 1, right. The system is again described by two coordinates  $(\mathbf{q}, \mathbf{u}) = (\theta, r) \in S^1 \times I$ . However, in this case, the upright equilibrium position is *not* stabilizable by any oscillatory motion of the radius  $r(t)$  around a fixed value.

A major difference between these two systems is that the equation of motion of the first one contains a quadratic term in the time derivative  $\dot{\mathbf{u}} \doteq d\mathbf{u}/dt$ . On the other hand, the equation for the variable-length pendulum is affine w.r.t. the variable  $\dot{\mathbf{u}}$ . Actually, the explicit dependence on  $\dot{\mathbf{u}}$  can be here entirely removed by a suitable change of coordinates.

To understand the general problem, one has to consider two main issues. The former is **geometric**, and involves the *orthogonal curvature* of the foliation

$$\Lambda \doteq \left\{ \mathcal{Q} \times \{\mathbf{u}\}, \quad \mathbf{u} \in \mathcal{U} \right\}. \quad (1.1)$$

Orthogonality is here defined w.r.t. the Riemannian metric determined by the kinetic energy. The orthogonal curvature is a measure of how a geodesic, which is perpendicular to the leaf  $\mathcal{Q} \times \{\mathbf{u}\}$  of the foliation at a given point  $(\mathbf{q}, \mathbf{u})$ , fails to remain perpendicular to the other leaves it meets. If this curvature is non-zero, then the dynamic equations for  $\mathbf{q}$  and for the corresponding momentum  $\mathbf{p}$  contain a quadratic term in the time derivative  $\dot{\mathbf{u}}$  of the control function. This will be analyzed in detail in Part I, Sections 5, 6.

The latter issue is **analytical**, namely: how to exploit this curvature, i.e. the quadratic terms

in  $\dot{\mathbf{u}}$ , in order to achieve stabilization. This will be discussed in Part II of this paper. In particular, we study the set of solutions for a system with quadratic, unbounded, controls, making essential use of reparametrization techniques. These, in turn, are combined with arguments involving Lyapunov functions for a convexified system.

The paper consists of three parts. In order to keep our exposition as self-contained as possible, in Part I we first describe the mechanical model and derive the basic dynamical equations. In Section 2 we recall the classical equations of motion, in a Hamiltonian intrinsic form, for time-dependent holonomic systems subject to non-conservative forces. Although this is a classical subject, which can be found in many text-books in Classical Mechanics, the purpose of this first section is to clarify concepts and notations used in the remainder of the paper. In Section 3 we consider a state space  $\mathcal{Y} = \mathcal{Q} \times \mathcal{U}$  given by the product of two manifolds. The *controls* will be curves  $t \mapsto \mathbf{u}(t)$  taking values in the manifold  $\mathcal{U}$ . The main physical assumption we are making is that these controls  $\mathbf{u}(\cdot)$  are implemented by means of frictionless, time-dependent constraints. One can then derive the equations of motion on the reduced state space  $\mathcal{Q}$ , where the dynamics depends on  $\mathbf{u}$  and on its time derivative  $\dot{\mathbf{u}}$ , the latter dependence being polynomial of degree two. In Section 4, we deduce the local expression of the control equations in a system of local coordinates adapted to the foliation  $\Lambda$  in (1.1). Section 5 contains a survey of some geometrical and functional analytic results concerning the input-output map and the kinetic metric. The main new result of Part I appears in Section 6, where we present a new interpretation of the quadratic dependence of the equations of motion on the derivative of the control functions. Our characterization of the quadratic coefficients is given in terms of the concatenation of two geodesics, the second returning to the same leaf of the foliation where the first one had started. This generalizes to higher dimensions a result in [16], where the scalar control case is considered. Finally, Section 7 provides a variational characterization of admissible control-trajectory pairs.

In Part II we consider a general nonlinear system where the right hand side is a quadratic polynomial w.r.t. the time derivatives of the control function.

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha,\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}. \quad (1.2)$$

Using a re-parametrization technique, we show that the stabilization problem for the impulsive control system (1.2) can be reduced to proving a weak stability property for a related differential inclusion with compact, convex-valued right hand side:

$$\frac{d}{ds}x(s) \in F(x(s)), \quad (1.3)$$

$$F(x) \doteq \overline{\text{co}} \left\{ f(x) w_0^2 + \sum_{\alpha=1}^m g_{\alpha}(x) w_0 w_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha,\beta}(x) w_{\alpha} w_{\beta} ; \quad w_0 \in [0, 1], \quad \sum_{\alpha=0}^m w_{\alpha}^2 = 1 \right\},$$

where  $\overline{\text{co}}$  denotes a closed convex hull. Theorems 9.1 and 9.2 relate the weak (asymptotic) stabilizability of the differential inclusion (1.3) with the (asymptotic) stabilizability of the impulsive control system (1.2).

In practical cases, a direct analysis of the multifunction  $F$  may be difficult. In Section 9, in addition to (1.3) we thus consider an auxiliary differential inclusion of the form  $\dot{x} \in G(x)$ , where the multifunction  $G$  is derived from (1.2) by neglecting all linear terms, i.e. by formally

setting  $g_\alpha \equiv 0$ . We show that the weak stability of this second differential inclusion still yields the relevant stabilization properties for the original control system (1.2). Motivated by [29], in Section 10 we also show that the weak stability of the differential inclusion can be established by looking at suitable selections.

In Part III we apply the previous analytic results to the problem of stabilization of mechanical systems, controlled by moving holonomic constraints. Thanks to the particular structure of the quadratic terms that appear in the equations of motion, we show that in many cases one can construct a suitable Lyapunov function, and thus establish the desired stability properties. The paper is then concluded with some examples, presented in Section 12.

Throughout the paper, our focus is on systems in general form, where the equations of motion depend quadratically on the time derivatives  $\dot{u}_\alpha$ . In the special case where the dependence is only linear, i.e.  $h_{\alpha,\beta} \equiv 0$  in (1.2), our results still apply; however, controllability and stabilization are best studied by looking at Lie brackets of the vector fields  $f, g_\alpha$ , using standard techniques of geometric control theory [13, 32].

In addition to [8, 19], readers interested in the earlier developments of the theory of control of mechanical systems by moving constraints are referred to [9, 12, 10, 22, 23]. A concise survey, also outlining possible applications to swim-like motion in fluids, has recently appeared in [3]. See also the lecture notes in [25].

## Part I

# Time-dependent holonomic constraints as controls

## 2 Review of the dynamical equations on the cotangent bundle

### THE LEGENDRE-FENCHEL TRANSFORM

Let  $W$  be a finite-dimensional, real vector space, let  $W^*$  be its dual space, and let  $\langle \cdot, \cdot \rangle$  denote the duality between  $W$  and  $W^*$ . For every map  $L : W \mapsto \mathbb{R}$ , its Legendre-Fenchel transform is defined as

$$L^*(\mathbf{P}) \doteq \sup_{\mathbf{V} \in W} \{ \langle \mathbf{P}, \mathbf{V} \rangle - L(\mathbf{V}) \}, \quad (2.1)$$

for every  $\mathbf{P} \in W^*$ .  $L^*$  is a convex map. If  $L$  is strictly convex, then  $L^*$  is strictly convex, and  $(L^*)^* = L$ .

In the special case where  $L$  is the sum of a positive definite quadratic form and an affine function, the same is true of its transform  $L^*$ . Moreover, the following facts are well-known:

- For every  $\mathbf{P} \in W^*$  there exists a unique  $\mathbf{V} = i^L(\mathbf{P}) \in W$  where the maximum on the right-hand side of (2.1) is achieved. The map  $i^L : W^* \mapsto W$  is one-to-one, affine, and satisfies

$$(i^L)^{-1} = i^{L^*}. \quad (2.2)$$

If  $L$  is a quadratic form then  $i^L$  and  $i^{L^*}$  are linear.

- By identifying both  $W$  and  $W^*$  with  $\mathbb{R}^n$  (by the choice of a basis on  $W$  and of the dual basis on  $W^*$ ) and using  $\frac{\partial L}{\partial \mathbf{V}}$  and  $\frac{\partial L^*}{\partial \mathbf{P}}$  to denote the gradients of  $L$  and  $L^*$ , respectively, one has

$$i^L(\mathbf{P}) = \frac{\partial L^*}{\partial \mathbf{P}}(\mathbf{P}) \quad (i^L)^{-1}(\mathbf{V}) = i^{L^*}(\mathbf{V}) = \frac{\partial L}{\partial \mathbf{V}}(\mathbf{V}).$$

## HOLONOMIC MECHANICAL SYSTEMS

Let  $\mathcal{Y}$  be a  $d$ -dimensional manifold, and call  $T\mathcal{Y} \doteq \{T_{\mathbf{y}}; \mathbf{y} \in \mathcal{Y}\}$ ,  $T^*\mathcal{Y} \doteq \{T_{\mathbf{y}}^*; \mathbf{y} \in \mathcal{Y}\}$  its tangent and cotangent bundles. If  $\mathcal{W} \subseteq \mathcal{Y}$  is an open subset, and  $Y : \mathcal{W} \mapsto \mathbb{R}^d$  is a coordinate chart, we recall that the corresponding *bundle charts*  $(Y, V)$  and  $(Y, P)$  on  $T\mathcal{Y}$  and  $T^*\mathcal{Y}$ , are obtained by choosing the local frames  $\left\{\frac{\partial}{\partial Y_r}; 1 \leq r \leq d\right\}$  and  $\{dY_r; 1 \leq r \leq d\}$ , respectively.

By a *holonomic mechanical system* with  $d$  degrees of freedom defined on a time interval  $I \subset \mathbb{R}$  we mean a pair  $\Sigma = (\mathcal{Y}, \mathcal{T})$ , where:

- $\mathcal{Y}$  is a  $d$ -dimensional manifold;
- $\mathcal{T} : I \times T\mathcal{Y} \mapsto \mathbb{R}$  is a map, called the *kinetic energy of the system*  $\Sigma$ , such that  $\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2$ , with

$$\begin{aligned} \mathcal{T}_0(t, \mathbf{y}, \mathbf{V}) &\doteq \frac{1}{2} \mathbf{g}_0(t, \mathbf{y}), & \mathcal{T}_1(t, \mathbf{y}, \mathbf{V}) &\doteq \langle \mathbf{g}_1(t, \mathbf{y}), \mathbf{V} \rangle, \\ \mathcal{T}_2(t, \mathbf{y}, \mathbf{V}) &\doteq \frac{1}{2} \mathbf{g}_2(t, \mathbf{y})(\mathbf{V}, \mathbf{V}). \end{aligned} \tag{2.3}$$

In (2.3), the  $\mathbf{g}_i$  are smooth maps defined on  $I \times \mathcal{Y}$ . For each  $(t, \mathbf{y})$ , the map  $\mathbf{g}_2(t, \mathbf{y})$  is a positive-definite quadratic form,  $\mathbf{g}_1(t, \mathbf{y})$  is linear map, while  $\mathbf{g}_0(t, \mathbf{y})$  is a constant.

We say that the system  $\Sigma = (\mathcal{Y}, \mathcal{T})$  is *time-independent* if  $\mathbf{g}_1(t, \mathbf{y}) = 0$ ,  $\mathbf{g}_0(t, \mathbf{y}) = 0$  for all  $(t, \mathbf{y}) \in I \times \mathcal{Y}$ , and  $\mathbf{g}_2$  is independent of  $t$ . Equivalently,  $\mathcal{T} = \mathcal{T}_2$  and does not depend on time. In the frame-work of Lagrangian mechanics, this means that the holonomic constraints determining the system  $\Sigma$  are time-independent. In this case, the notion of mechanical system coincides with that of Riemannian manifold, endowed with the metric  $\mathbf{g}_2$ .

In terms of a local bundle chart  $(Y, V)$  on  $T\mathcal{Y}$ , the maps in (2.3) take the form

$$\mathcal{T}_0(t, Y, V) = \frac{1}{2} g_0, \quad \mathcal{T}_1(t, Y, V) = g_r V^r, \quad \mathcal{T}_2(t, Y, V) = \frac{1}{2} g_{rs} V^r V^s. \tag{2.4}$$

Here the  $d \times d$  matrix  $(g_{r,s})$ , the row vector  $(g_r)$ , and the real number  $g_0$  are the coordinate representations of  $\mathbf{g}_2(t, \mathbf{y})$ ,  $\mathbf{g}_1(t, \mathbf{y})$ , and  $\mathbf{g}_0(t, \mathbf{y})$ , respectively. Here and in the sequel, it is understood that a summation should be performed over repeated indices.

Let us define the map  $\mathcal{T}^* : I \times T^*\mathcal{Y} \mapsto \mathbb{R}$  as the Legendre transform of  $\mathcal{T}$ . For every  $(t, \mathbf{y}) \in I \times \mathcal{Y}$ , this means

$$\mathcal{T}^*(t, \mathbf{y}, \mathbf{P}) = \sup_{\mathbf{V} \in T_{\mathbf{y}}\mathcal{Y}} \{ \langle \mathbf{P}, \mathbf{V} \rangle - \mathcal{T}(t, \mathbf{y}, \mathbf{V}) \}.$$

Similarly, we define  $\mathcal{T}_2^*$  as the Legendre transform of  $\mathcal{T}_2$ .

The maps  $\mathcal{T}^*$  and  $\mathcal{T}_2^*$  will be also called the *kinetic Hamiltonians* corresponding to  $\mathcal{T}$  and  $\mathcal{T}_2$ , respectively. Accordingly, we shall use the notation

$$\mathcal{H} \doteq \mathcal{T}^* \quad \mathcal{H}_2 \doteq \mathcal{T}_2^*.$$

As in (2.2), for every  $(t, \mathbf{y}) \in I \times \mathcal{Y}$  we have the affine isomorphisms  $i_{t, \mathbf{y}}^{\mathcal{T}} : T_{\mathbf{y}}^* \mathcal{Y} \mapsto T_{\mathbf{y}} \mathcal{Y}$  and  $i_{t, \mathbf{y}}^{\mathcal{T}^*} : T_{\mathbf{y}} \mathcal{Y} \mapsto T_{\mathbf{y}}^* \mathcal{Y}$ , defined as

$$i_{t, \mathbf{y}}^{\mathcal{T}} \doteq i^{\mathcal{T}(t, \mathbf{y}, \cdot)} \quad \text{and} \quad i_{t, \mathbf{y}}^{\mathcal{T}^*} \doteq i^{\mathcal{T}^*(t, \mathbf{y}, \cdot)}. \quad (2.5)$$

respectively. Entirely similar linear isomorphisms  $i_{t, \mathbf{y}}^{\mathcal{T}_2}$  and  $i_{t, \mathbf{y}}^{\mathcal{T}_2^*}$  can be defined in connection with the quadratic map  $\mathcal{T}_2$ . According to (2.2), for all  $(t, \mathbf{y}) \in I \times \mathcal{Y}$  one has

$$\left(i_{t, \mathbf{y}}^{\mathcal{T}}\right)^{-1} = i_{t, \mathbf{y}}^{\mathcal{T}^*} \quad \left(i_{t, \mathbf{y}}^{\mathcal{T}_2}\right)^{-1} = i_{t, \mathbf{y}}^{\mathcal{T}_2^*}.$$

Using coordinates, if we use  $(g^{r,s})$  to denote the inverse of the  $d \times d$  matrix  $(g_{r,s})$  in (2.4), then the affine isomorphisms  $i_{t, \mathbf{y}}^{\mathcal{T}}$  and  $i_{t, \mathbf{y}}^{\mathcal{T}^*} = \left(i_{t, \mathbf{y}}^{\mathcal{T}}\right)^{-1}$  are given by

$$V^s = i_{t, \mathbf{y}}^{\mathcal{T}}(P_1, \dots, P_d) \doteq g^{r,s}(P_r - g_r) \quad s = 1, \dots, d, \quad (2.6)$$

$$P_s = i_{t, \mathbf{y}}^{\mathcal{T}^*}(V^1, \dots, V^d) \doteq g_{r,s}V^r + g_s \quad s = 1, \dots, d. \quad (2.7)$$

By the identity

$$\mathcal{H}(t, \mathbf{y}, \mathbf{P}) = \mathcal{T}^*(t, \mathbf{y}, \mathbf{P}) = \langle \mathbf{P}, i_{t, \mathbf{y}}^{\mathcal{T}}(\mathbf{P}) \rangle - \mathcal{T}(t, \mathbf{y}, i_{t, \mathbf{y}}^{\mathcal{T}}(\mathbf{P})),$$

it is straightforward to check that the Hamiltonian can be decomposed into a constant, a linear, and a quadratic part. Namely,  $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2$  with

$$\mathcal{H}_2(t, \mathbf{y}, \mathbf{P}) = \frac{1}{2} \mathbf{g}^2(t, \mathbf{y})(\mathbf{P}, \mathbf{P}) = \frac{1}{2} \mathbf{g}_2(t, \mathbf{y}) \left( i_{t, \mathbf{y}}^{\mathcal{T}_2}(\mathbf{P}), i_{t, \mathbf{y}}^{\mathcal{T}_2}(\mathbf{P}) \right),$$

$$\mathcal{H}_1(t, \mathbf{y}, \mathbf{P}) = \mathbf{g}^1(t, \mathbf{y})(\mathbf{P}) = - \langle \mathbf{g}_1(t, \mathbf{y}), i_{t, \mathbf{y}}^{\mathcal{T}_2}(\mathbf{P}) \rangle,$$

$$\mathcal{H}_0(t, \mathbf{y}) = \frac{1}{2} \mathbf{g}^0(t, \mathbf{y}) = \frac{1}{2} \left[ \mathbf{g}^2(t, \mathbf{y}) \left( \mathbf{g}_1(t, \mathbf{y}), \mathbf{g}_1(t, \mathbf{y}) \right) - \mathbf{g}_0(t, \mathbf{y}) \right].$$

We remark that, by our assumptions, the quadratic form  $\mathbf{g}^2(t, \mathbf{y})$  is positive definite.

Using local coordinates  $(Y, P)$  the decomposition of the Hamiltonian function takes the form  $\mathcal{H}(t, Y, P) = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2$ , where

$$\mathcal{H}_0 \doteq \frac{1}{2} g^{r,s} g_r g_s - \frac{g_0}{2}, \quad \mathcal{H}_1 \doteq -g^{r,s} P_r g_s, \quad \mathcal{H}_2 \doteq \frac{1}{2} g^{r,s} P_r P_s. \quad (2.8)$$

SYMPLECTIC STRUCTURE

Consider again the  $d$ -dimensional manifold  $\mathcal{Y}$ . By  $\omega_{\mathcal{Y}}$  we shall denote the canonical symplectic form on  $T^*\mathcal{Y}$ . This is the (nondegenerate) 2-form which, in any chart of  $\mathcal{Y}$  and the corresponding bundle chart of  $T^*\mathcal{Y}$ , is represented by the constant  $2d \times 2d$  symplectic matrix

$$S_d \doteq \begin{pmatrix} 0_d & \mathbf{I}_d \\ -\mathbf{I}_d & 0_d \end{pmatrix}.$$

For each  $(\mathbf{y}, \mathbf{P}) \in T^*\mathcal{Y}$ , by means of the canonical symplectic form  $\omega_{\mathcal{Y}}$  one can establish a linear isomorphism  $s_{\mathbf{y}, \mathbf{P}} : T_{\mathbf{y}, \mathbf{P}}^*(T^*\mathcal{Y}) \mapsto T_{\mathbf{y}, \mathbf{P}}(T^*\mathcal{Y})$ . This is uniquely defined by setting

$$\langle \beta, \mathbf{w} \rangle = \omega_{\mathcal{Y}}(\mathbf{y}, \mathbf{P}) (s_{\mathbf{y}, \mathbf{P}}(\beta), \mathbf{w}). \quad (2.9)$$

With reference to coordinates  $(Y, P)$ , if  $\beta = (\beta_r dY^r + \gamma_r dP_r)$ , one has

$$s_{\mathbf{y}, \mathbf{P}}(\beta) = \gamma_r \frac{\partial}{\partial Y^r} - \beta_r \frac{\partial}{\partial P_r},$$

For every function  $\phi : I \times T^*\mathcal{Y} \mapsto \mathbb{R}$ , differentiable w.r.t. the second variable for every  $t \in I$ , one can define a time-dependent vector field  $X_\phi$  on  $T^*\mathcal{Y}$ , called the *Hamiltonian vector field* associated to  $\phi$ . For each  $(t, \mathbf{y}, \mathbf{P}) \in I \times T^*\mathcal{Y}$ , this is defined by

$$X_\phi(t, \mathbf{y}, \mathbf{P}) \doteq s_{\mathbf{y}, \mathbf{P}}(d\phi(t, \mathbf{y}, \mathbf{P}))$$

where  $d\phi(t, \cdot)$  denotes the differential of the map  $\phi(t, \cdot) : T^*\mathcal{Y} \mapsto \mathbb{R}$ . In local coordinates, one has

$$X_\phi = \frac{\partial \phi}{\partial P_r} \frac{\partial}{\partial Y^r} - \frac{\partial \phi}{\partial Y^r} \frac{\partial}{\partial P_r}.$$

## FORCES

Forces are represented by vertical vector fields on the cotangent bundle  $T^*\mathcal{Y}$ . We recall that, for  $(\mathbf{y}, \mathbf{P}) \in T^*\mathcal{Y}$ , a vector  $X \in T_{(\mathbf{y}, \mathbf{P})}(T^*\mathcal{Y})$  is called *vertical* if

$$d\Pi(\mathbf{y}, \mathbf{P})(X) = 0.$$

Here  $\Pi : T^*\mathcal{Y} \mapsto \mathcal{Y}$  is the canonical projection of  $T^*\mathcal{Y}$  on  $\mathcal{Y}$  and  $d\Pi(\mathbf{y}, \mathbf{P})$  denotes its differential at  $(\mathbf{y}, \mathbf{P})$ . The subspace of vertical vectors at  $(\mathbf{y}, \mathbf{P})$  is thus the kernel of the map  $d\Pi(\mathbf{y}, \mathbf{P})$ . It will be called the **vertical tangent space** of  $T^*\mathcal{Y}$  at  $(\mathbf{y}, \mathbf{P})$  and denoted by  $V_{(\mathbf{y}, \mathbf{P})}(T^*\mathcal{Y})$ . The corresponding fiber sub-bundle of  $T(T^*\mathcal{Y})$  is called the *vertical tangent bundle* of  $T^*\mathcal{Y}$  and denoted by  $V(T^*\mathcal{Y})$ .

A vector field  $\mathbf{F}$  on  $(T^*\mathcal{Y})$  is said to be *vertical* if  $\mathbf{F}(\mathbf{y}, \mathbf{P}) \in V_{(\mathbf{y}, \mathbf{P})}(T^*\mathcal{Y})$  for each  $(\mathbf{y}, \mathbf{P}) \in T^*\mathcal{Y}$ . Using a canonical bundle chart of  $T^*\mathcal{Y}$ , a vertical vector field is thus represented by a  $2d$ -dimensional column vector of the form  $(0, \dots, 0, F_1, \dots, F_d)^\dagger$ , where the superscript  $^\dagger$  denotes transposition.

More generally, let  $I$  be a real interval. A function  $\mathbf{F} : I \times T^*\mathcal{Y} \mapsto V(T^*\mathcal{Y})$  such that, for each  $t \in I$ , the map  $\mathbf{F}(t, \cdot)$  is a vertical vector field on  $T^*\mathcal{Y}$  will be called a *time-dependent*



*vertical vector field*. In order to retain its physical meaning,  $\mathbf{F}$  will also be called a (possibly time-dependent) *force* acting on  $\mathcal{Y}$  during the time-interval  $I$ .

A time-independent force  $\mathbf{F}$  is called *positional* if  $\mathbf{F}(\mathbf{y}, \mathbf{P}) = \mathbf{F}(\mathbf{y})$ , i.e. if its values depend only on  $\mathbf{y}$  and not on the value  $\mathbf{P} \in T_{\mathbf{y}}$ . Furthermore, a positional force  $\mathbf{F}$  is called *conservative* if there exists a potential function  $U : \mathcal{Y} \mapsto \mathbb{R}$  such that  $\mathbf{F}$  is the Hamiltonian vector field corresponding to  $U$ , namely

$$\mathbf{F} = X_U. \quad (2.10)$$

Recalling the symplectic form  $\omega_{\mathcal{Y}}$  in (2.9), we now introduce a formal notion of *power* of the force  $\mathbf{F}$ , which will play an important role in Section 3. Let  $(\mathbf{y}, \mathbf{P}) \in T^*\mathcal{Y}$ ,  $\mathbf{F} \in V_{(\mathbf{y}, \mathbf{P})}(T^*\mathcal{Y})$ , and  $\mathbf{V} \in T(T^*\mathcal{Y})$ . The quantity

$$\omega_{\mathcal{Y}}(\mathbf{F}, \mathbf{V}) \quad (2.11)$$

will be called the *power of  $\mathbf{F}$  with respect to  $\mathbf{V}$* . If  $(Y, P)$  are canonical local coordinates on  $T^*\mathcal{Y}$  and

$$\mathbf{F} = F_r \frac{\partial}{\partial P_r} \quad \mathbf{V} = V_r \frac{\partial}{\partial Y^r} + W_r \frac{\partial}{\partial P_r},$$

then one obtains the familiar expression for the power:

$$\omega_{\mathcal{Y}}(\mathbf{F}, \mathbf{V}) = F_r V_r.$$

## THE EQUATION OF MOTION

Let  $\Sigma = (\mathcal{Y}, \mathcal{T})$  be a mechanical system and let  $\mathcal{H}$  be the corresponding kinetic Hamiltonian. Let  $\mathbf{F}$  be a force acting on  $\mathcal{Y}$ . Then the **equation of motion** for the mechanical system  $\Sigma$  subject to the force  $\mathbf{F}$  is the differential equation

$$\frac{d}{dt} \begin{pmatrix} \mathbf{y} \\ \mathbf{P} \end{pmatrix} = X_{\mathcal{H}} + \mathbf{F} \quad t \in I, \quad (\mathbf{y}, \mathbf{P}) \in T^*\mathcal{Y}, \quad (2.12)$$

where  $X_{\mathcal{H}}$  is the Hamiltonian vector field associated to  $\mathcal{H}$ .

If  $\mathbf{F}$  is a conservative force with potential function  $U : \mathcal{Y} \mapsto \mathbb{R}$ , we can consider the standard Hamiltonian

$$H \doteq (\mathcal{T} - U)^*,$$

defined as the Legendre transform of  $\mathcal{T} - U$ . Then (2.12) reduces to the usual Hamiltonian form

$$\frac{d}{dt} \begin{pmatrix} \mathbf{y} \\ \mathbf{P} \end{pmatrix} = X_H \quad t \in I. \quad (2.13)$$

Indeed, one has  $X_{\mathcal{H}} + X_U = X_{\mathcal{H}+U}$  and  $\mathcal{H} + U = (\mathcal{T} - U)^* = H$ .

With the usual notational conventions, the equation of motion (2.12) in a local bundle chart takes the form

$$\begin{cases} \dot{Y}^r = \frac{\partial \mathcal{H}}{\partial P_r} \\ \dot{P}_r = -\frac{\partial \mathcal{H}}{\partial Y^r} + F_r \end{cases} \quad r = 1, \dots, d. \quad (2.14)$$

Using the expressions (2.8) for the Hamiltonian, from (2.14) we obtain

$$\begin{cases} \dot{Y}^r = g^{r,s} (P_s - g_s), \\ \dot{P}_r = -\frac{\partial g^{\ell s}}{\partial Y^r} \left( \frac{1}{2} g_\ell g_s - P_\ell g_s + \frac{1}{2} P_\ell P_s \right) + g^{\ell s} \frac{\partial g_s}{\partial Y^r} (P_\ell - g_\ell) + \frac{\partial q_0}{\partial Y^r} + F_r. \end{cases} \quad (2.15)$$

In the case of a time-independent system, these equations reduce to

$$\begin{cases} \dot{Y}^r = g^{r,s} P_s, \\ \dot{P}_r = -\frac{1}{2} \frac{\partial g^{\ell s}}{\partial Y^r} P_\ell P_s + F_r. \end{cases} \quad (2.16)$$

In particular, if the force is conservative, (2.16) takes the familiar Hamiltonian form

$$\begin{cases} \dot{Y}^r = \frac{\partial H}{\partial P_r} & (= g^{r,s} P_s), \\ \dot{P}_r = -\frac{\partial H}{\partial Y^r} & (= -\frac{1}{2} \frac{\partial g^{\ell s}}{\partial Y^r} P_\ell P_s - \frac{\partial U}{\partial Y^r}). \end{cases} \quad (2.17)$$

### 3 Time-dependent constraints as controls

In this section we shall set up the general framework to treat the situation where additional time-dependent holonomic constraints are regarded as *controls*.

#### 3.1 Structural assumptions

We shall consider a mechanical system  $\Sigma = (\mathcal{Y}, \mathcal{T})$  verifying the following assumptions:

- 1) (PRODUCT STRUCTURE). The state manifold  $\mathcal{Y}$  is a product manifold, namely

$$\mathcal{Y} = \mathcal{Q} \times \mathcal{U}, \quad (3.1)$$

Here  $\mathcal{Q}$  and  $\mathcal{U}$ , called the *reduced state space* and the *control space*, are manifolds of dimension  $N$  and  $M$ , respectively.

- 2) (STATIONARITY OF THE METRIC)<sup>1</sup> The Lagrangian system  $\Sigma$  is time-independent. Namely, the kinetic energy  $\mathcal{T}$  is defined by a triple  $\mathbf{g} = (\mathbf{g}_2, 0, 0)$  with  $\mathbf{g}_2$  independent of time.
- 3) (REGULARITY OF THE FORCE). The external force  $\mathbf{F} = \mathbf{F}(t, \mathbf{q}, \mathbf{u}, \mathbf{P}, \varphi)$  is a function measurable w.r.t.  $t$  and locally Lipschitz w.r.t. all other variables.

By (3.1), one has the natural identifications of  $T^*(\mathcal{Q} \times \mathcal{U})$ ,  $T(T^*(\mathcal{Q} \times \mathcal{U}))$ , and  $V(T^*(\mathcal{Q} \times \mathcal{U}))$  with the products  $T^*(\mathcal{Q}) \times T^*(\mathcal{U})$ ,  $T(T^*(\mathcal{Q})) \times T(T^*(\mathcal{U}))$ , and  $V(T^*(\mathcal{Q})) \times V(T^*(\mathcal{U}))$ , respectively.

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<sup>1</sup>This assumption is made only for the sake of simplification, since it avoids a double time-dependence: the structural one and the one due to the implementation of controls. Actually, the situation where  $\mathbf{g}$  has a general form can be treated as well without significative additional difficulties.

By the second assumption,  $\mathcal{T} = \mathcal{T}_2$ , and, for every  $(t, \mathbf{y}) \in I \times \mathcal{Y}$ , the affine isomorphism  $i_{t, \mathbf{y}}^{\mathcal{T}} = i_{\mathbf{y}}^{\mathcal{T}}$  is a linear isomorphism, independent of time. In turn the Hamiltonian is time-independent and its global and local expressions are given by

$$\mathcal{H}(\mathbf{y}, \mathbf{P}) = \frac{1}{2} \mathbf{g}^{-1}(\mathbf{y})(\mathbf{P}, \mathbf{P})$$

and

$$\mathcal{H}(Y, P) = \frac{1}{2} \sum_{r,s=1}^{N+M} g^{r,s}(Y) P_r P_s ,$$

respectively. Moreover, for each  $\mathbf{y} \in \mathcal{Y}$ , as soon as the vector spaces  $T_{\mathbf{y}}\mathcal{Y}$  and  $T_{\mathbf{y}}^*\mathcal{Y}$  are endowed with the scalar products defined by  $\mathbf{g}(\mathbf{y})$  and  $\mathbf{g}^{-1}(\mathbf{y})$ , respectively, the isomorphism  $i_{\mathbf{y}}^{\mathcal{T}}$  is an isometry. In particular, one has

$$\mathcal{T}(\mathbf{y}, i_{\mathbf{y}}^{\mathcal{T}}(\mathbf{P})) = \mathcal{H}(\mathbf{y}, \mathbf{P})$$

for all  $\mathbf{y} \in \mathcal{Y}$  and  $\mathbf{P} \in T_{\mathbf{y}}^*\mathcal{Y}$ .

We observe that, for every  $(\mathbf{q}, \mathbf{u}) \in \mathcal{Q} \times \mathcal{U}$ , the map  $i_{\mathbf{q}, \mathbf{u}}^{\mathcal{T}}$  can be naturally split in two components. Indeed, for  $(\mathbf{p}, \wp) \in T_{\mathbf{q}}^*\mathcal{Q} \times T_{\mathbf{u}}^*\mathcal{U}$ , we can write

$$i_{\mathbf{q}, \mathbf{u}}^{\mathcal{T}}(\mathbf{p}, \wp) = \left( (i_{\mathbf{q}, \mathbf{u}}^{\mathcal{T}})^{\mathcal{Q}}(\mathbf{p}, \wp), (i_{\mathbf{q}, \mathbf{u}}^{\mathcal{T}})^{\mathcal{U}}(\mathbf{p}, \wp) \right) \in T_{\mathbf{q}}\mathcal{Q} \times T_{\mathbf{u}}\mathcal{U}. \quad (3.2)$$

### 3.2 Foliation structure and adapted coordinates

The product structure of  $\mathcal{Y} = \mathcal{Q} \times \mathcal{U}$  induces a foliation structure, where the set of leaves is

$$\Lambda = \{ \mathcal{Q} \times \{\mathbf{u}\} \mid \mathbf{u} \in \mathcal{U} \}. \quad (3.3)$$

For every  $(\mathbf{q}, \mathbf{u}) \in \mathcal{Q} \times \mathcal{U}$ , we denote by  $\Lambda(\mathbf{q}, \mathbf{u}) \doteq \mathcal{Q} \times \{\mathbf{u}\}$  the *leaf* through  $(\mathbf{q}, \mathbf{u})$ . Let us consider the corresponding distribution<sup>2</sup>  $\Delta$ , whose fibers are given by

$$\Delta_{(\mathbf{q}, \mathbf{u})} = T_{\mathbf{q}}\mathcal{Q} \times \{0\}.$$

In our analysis, a very important role will also be played by the *orthogonal* distribution

$$\Delta_{(\mathbf{q}, \mathbf{u})}^{\perp} = \left\{ Y \in T_{\mathbf{q}}\mathcal{Q} \times T_{\mathbf{u}}\mathcal{U} \mid \mathbf{g}(\mathbf{q}, \mathbf{u})(Y, X) = 0 \text{ for all } X \in \Delta_{(\mathbf{q}, \mathbf{u})} \right\}, \quad (3.4)$$

also called the *orthogonal bundle*, for short.

In connection with the foliation  $\Lambda$  at (3.3), we say that a system of coordinates  $(\tilde{q}, \tilde{u})$  is  $\Lambda$ -*adapted* if the sets  $\{\tilde{u} = \text{constant}\}$  locally coincide with the leaves of the foliation. Of course, the local product coordinates  $(q, u)$  are  $\Lambda$ -adapted. More generally, if  $(\tilde{q}, \tilde{u})$  are  $\Lambda$ -adapted, then every system of coordinates  $(\hat{q}, \hat{u})$  obtained from  $(\tilde{q}, \tilde{u})$  by means of a local diffeomorphism of the form

$$\hat{q} = \hat{q}(\tilde{q}, \tilde{u}) \quad \hat{u} = \hat{u}(\tilde{u}). \quad (3.5)$$

is  $\Lambda$ -adapted as well.

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<sup>2</sup>In our context, the term “distribution” is meant in the sense of differential geometry, namely, a fiber sub-bundle of the tangent bundle  $T(\mathcal{Q} \times \mathcal{U})$ .

### 3.3 Admissible input-output pairs

Consider a control function  $t \mapsto \mathbf{u}(t) \in \mathcal{U}$ . In this section we characterize the corresponding output  $t \mapsto (\mathbf{q}(t), \mathbf{p}(t))$  as the solution of a certain Cauchy problem. In the following section, we then show that our definition is consistent with the mechanical model, where the control is implemented in terms of frictionless constraints.

For every  $(\mathbf{u}, \mathbf{w}) \in T\mathcal{U}$ , let us define the map  $\mathcal{T}^{\mathbf{u}, \mathbf{w}} : T\mathcal{Q} \mapsto \mathbb{R}$  by setting

$$\mathcal{T}^{\mathbf{u}, \mathbf{w}}(\mathbf{q}, \mathbf{v}) \doteq \mathcal{T}(\mathbf{q}, \mathbf{u}, \mathbf{v}, \mathbf{w}), \quad (3.6)$$

for all  $(\mathbf{q}, \mathbf{v}) \in T\mathcal{Q}$ . This map can be regarded as the kinetic energy of the reduced system when the control takes the value  $\mathbf{u}$ , with  $\dot{\mathbf{u}} = \mathbf{w}$ .

Let  $I \subset \mathbb{R}$  be an interval, and let  $\mathbf{u} : I \mapsto \mathcal{U}$  be an absolutely continuous control. The (time-dependent) kinetic energy of the reduced system on  $\mathcal{Q}$ , corresponding to the control  $\mathbf{u}(\cdot)$  is described, almost every  $t \in I$  and for all  $(\mathbf{q}, \mathbf{v}) \in T\mathcal{Q}$ , by

$$(t, \mathbf{q}, \mathbf{v}) \mapsto \mathcal{T}^{\mathbf{u}(t), \dot{\mathbf{u}}(t)}(\mathbf{q}, \mathbf{v}).$$

The corresponding (time-dependent) Hamiltonian on  $T^*\mathcal{Q}$  is

$$(t, \mathbf{q}, \mathbf{p}) \mapsto \mathcal{H}^{\mathbf{u}(t), \dot{\mathbf{u}}(t)}(\mathbf{q}, \mathcal{P}),$$

where

$$\mathcal{H}^{\mathbf{u}, \mathbf{w}}(\mathbf{q}, \mathbf{p}) \doteq \sup_{\mathbf{v} \in T_{\mathbf{q}}\mathcal{Q}} \left\{ \langle \mathbf{p}, \mathbf{v} \rangle - \mathcal{T}^{\mathbf{u}, \mathbf{w}}(\mathbf{q}, \mathbf{v}) \right\}. \quad (3.7)$$

Since  $\mathbf{g}$  is positive definite, for every  $(\mathbf{q}, \mathbf{u}) \in \mathcal{Q} \times \mathcal{U}$  and every  $\mathbf{p} \in T_{\mathbf{q}}^*\mathcal{Q}$ , the affine function

$$\wp \mapsto \left( i_{\mathbf{q}, \mathbf{u}}^T \right)^{\mathcal{U}}(\mathbf{p}, \wp) \in T_{\mathbf{q}}\mathcal{Q}$$

in (3.2) is invertible. Its inverse will be denoted by

$$\mathbf{w} \mapsto \wp_{(\mathbf{q}, \mathbf{u}, \mathbf{p})}(\mathbf{w}).$$

Let  $(q, u)$  be  $\Lambda$ -adapted coordinates, and let  $(q, u, p, \pi)$  are the corresponding bundle coordinates. Let  $(F_i, F_{N+\alpha})$  be the components of the force  $\mathbf{F}$ , so that

$$\mathbf{F} = F_i \frac{\partial}{\partial p_i} + F_{N+\alpha} \frac{\partial}{\partial \pi_\alpha}. \quad (3.8)$$

Recalling the dimensions of the manifolds  $\mathcal{Q}$  and  $\mathcal{U}$ , we here have  $i = 1, \dots, N$  and  $\alpha = 1, \dots, M$ . The Einstein summation convention is always used. In addition, we set

$$F_{\mathcal{Q}} \doteq F_i \frac{\partial}{\partial p_i}. \quad (3.9)$$

Notice that  $F_{\mathcal{Q}}$  is independent of the chosen  $\Lambda$ -adapted system of coordinates. For every  $(\mathbf{u}, \mathbf{w}) \in T\mathcal{U}$  and  $i = 1, \dots, N$ , we also define

$$F_i^{\mathbf{u}, \mathbf{w}}(t, \mathbf{q}, \mathbf{p}) \doteq F_i(t, \mathbf{q}, \mathbf{u}, \mathbf{p}, \wp_{(\mathbf{q}, \mathbf{u}, \mathbf{p})}(\mathbf{w})) \quad (3.10)$$

and

$$\mathbf{F}_{\mathcal{Q}}^{\mathbf{u}, \mathbf{w}}(t, \mathbf{q}, \mathbf{p}) \doteq F_{\mathcal{Q}}(t, \mathbf{q}, \mathbf{u}, \mathbf{p}, \wp_{(\mathbf{q}, \mathbf{u}, \mathbf{p})}(\mathbf{w})) = F_i(t, \mathbf{q}, \mathbf{u}, \mathbf{p}, \wp_{(\mathbf{q}, \mathbf{u}, \mathbf{p})}(\mathbf{w})) \frac{\partial}{\partial p_i}. \quad (3.11)$$

**Remark 3.1** Despite (3.6), in general one has

$$\mathcal{H}^{\mathbf{u}, \mathbf{w}}(\mathbf{q}, \mathbf{p}) \neq \mathcal{H}(\mathbf{q}, \mathbf{u}, \mathbf{p}, \wp_{(\mathbf{q}, \mathbf{u}, \mathbf{p})}(\mathbf{w})) .$$

The actual relation between these two functions will be illustrated in Lemma 4.1.

**Definition 3.1** Let  $I \subset \mathbb{R}$  be a time interval. Let

$$\mathbf{u} : I \mapsto \mathcal{U} \quad (\mathbf{q}, \mathbf{p}) : I \mapsto T^*\mathcal{Q}$$

be absolutely continuous maps. We say that  $(\mathbf{u}(\cdot), (\mathbf{q}, \mathbf{p})(\cdot))$  is an admissible input-output pair if  $(\mathbf{q}, \mathbf{p})$  is a Carathéodory solution of the control equation of motion

$$\frac{d}{dt}(\mathbf{q}(t), \mathbf{p}(t)) = X_{\mathcal{H}^{\mathbf{u}(t), \dot{\mathbf{u}}(t)}}(\mathbf{q}(t), \mathbf{p}(t)) + \mathbf{F}_{\mathcal{Q}}^{\mathbf{u}(t), \dot{\mathbf{u}}(t)}(\mathbf{q}(t), \mathbf{p}(t)) . \quad (3.12)$$

Here  $X_{\mathcal{H}^{\mathbf{u}, \dot{\mathbf{u}}}}$  denotes the Hamiltonian vector field corresponding to  $\mathcal{H}^{\mathbf{u}, \dot{\mathbf{u}}}$ , with respect to the symplectic structure on  $T^*\mathcal{Q}$ .

We recall that a Carathéodory solutions of an ODE  $\dot{x} = f(t, x)$  is an absolutely continuous function  $t \mapsto x(t)$  that satisfies the differential equation at a.e. time  $t$ . Given an initial data

$$(\mathbf{q}(\bar{t}), \mathbf{p}(\bar{t})) = (\bar{\mathbf{q}}, \bar{\mathbf{p}}) , \quad (3.13)$$

and an absolutely continuous control function  $t \mapsto \mathbf{u}(t)$ , the existence and uniqueness of a corresponding admissible output  $(\mathbf{q}(\cdot), \mathbf{p}(\cdot))$  can be obtained from standard ODE theory.

Depending on the geometrical properties of the metric  $\mathbf{g}$ , the regularity assumptions on the input  $\mathbf{u}$  and the output  $(\mathbf{q}, \mathbf{p})$  can be considerably weakened. This fact, discussed later on in the paper, is essential for both optimization and stabilization purposes.

### 3.4 Realization of controls as frictionless constraints.

To motivate the previous notion of input-output pair, we need to recall the notion of *frictionless constraint reaction* in the Hamiltonian framework.

Let  $Pr_1 : T^*(\mathcal{Q}) \times T^*(\mathcal{U}) \rightarrow T^*(\mathcal{Q})$  denote the canonical projection on the first factor, and, for every  $((\mathbf{q}, \mathbf{p}), (\mathbf{u}, \wp)) \in T^*(\mathcal{Q}) \times T^*(\mathcal{U})$ , let  $D(Pr_1)(\mathbf{q}, \mathbf{p}, \mathbf{u}, \wp)$  denote its derivative.

Let us consider the subspace of vertical vectors

$$R_{(\mathbf{q}, \mathbf{p}), (\mathbf{u}, \wp)}^{\mathcal{Q}} \doteq \left( V_{(\mathbf{q}, \mathbf{p})}(T^*\mathcal{Q}) \times V_{(\mathbf{u}, \wp)}(T^*\mathcal{U}) \right) \cap \ker(D(Pr_1)(\mathbf{q}, \mathbf{p}, \mathbf{u}, \wp)) .$$

It is straightforward to verify that

$$R_{(\mathbf{q}, \mathbf{p}), (\mathbf{u}, \wp)}^{\mathcal{Q}} = \{0\} \times V_{(\mathbf{u}, \wp)}(T^*\mathcal{U}) .$$

**Definition 3.2** The subspace  $R_{(\mathbf{q}, \mathbf{p}), (\mathbf{u}, \wp)}^{\mathcal{Q}}$  will be called the subspace of  $\mathcal{Q}$ -frictionless reactions at  $((\mathbf{q}, \mathbf{p}), (\mathbf{u}, \wp))$ . The corresponding vector bundle based on  $T^*\mathcal{Q} \times T^*\mathcal{U}$  will be called the vector bundle of  $\mathcal{Q}$ -frictionless reactions.

**Remark 3.2** In terms of the canonical form  $\omega_{\mathcal{Y}}$  on  $\mathcal{Y} = \mathcal{Q} \times \mathcal{U}$ ,  $R_{(\mathbf{q}, \mathbf{u}), (\mathbf{p}, \varphi)}^{\mathcal{Q}}$  can be characterized as the subspace of  $V_{(\mathbf{q}, \mathbf{p})}(T^*\mathcal{Q}) \times V_{(\mathbf{u}, \varphi)}(T^*\mathcal{U})$  made of those vectors  $\Phi$  such that

$$\omega_{\mathcal{Y}}(\Phi, V) = 0 \quad \text{for all } \mathbf{V} \in T_{(\mathbf{q}, \mathbf{p})}(T^*\mathcal{Q}) \times \{0\} \quad (3.14)$$

We remind that, in the language of symplectic geometry,  $\Phi$  and  $\mathbf{V}$  are also said *anti-orthogonal*.

**Remark 3.3** We are here regarding the constraint reactions as a set-valued force, described by the multifunction

$$(\mathbf{q}, \mathbf{p}), (\mathbf{u}, \varphi) \mapsto R_{(\mathbf{q}, \mathbf{p}), (\mathbf{u}, \varphi)}.$$

To check that this definition coincides with the usual one it is sufficient to notice that if  $\Phi \in R_{\mathbf{q}, \mathbf{p}, \mathbf{u}, \varphi}^{\mathcal{Q}}$  and  $\Phi = \sum_{r=1}^{N+M} \Phi_r \frac{\partial}{\partial P_r}$  is its local expression, then (3.14) is equivalent to

$$\langle (\Phi_1, \dots, \Phi_N, \Phi_{N+1}, \dots, \Phi_{N+M}), (v_1, \dots, v_N, 0, \dots, 0) \rangle = 0 \quad \text{for all } v_1, \dots, v_N \in \mathbb{R}.$$

Of course, this holds if and only if  $\Phi_i = 0$  for all  $i = 1, \dots, N$ .

Definition 3.1 is justified by Theorem 3.1 below. Let  $I \subset \mathbb{R}$  be an interval, and, for every  $(t, \mathbf{q}, \mathbf{u}, \mathbf{p}, \varphi) \in I \times T^*\mathcal{Q} \times T^*\mathcal{U} \rightarrow T^*\mathcal{Q}$ , let us set

$$X_{\mathcal{H}}^{\mathcal{Q}}(t, \mathbf{q}, \mathbf{u}, \mathbf{p}, \varphi) \doteq D(Pr_1) \cdot X_{\mathcal{H}}(t, \mathbf{q}, \mathbf{u}, \mathbf{p}, \varphi).$$

Notice that, according to (3.11),

$$F_{\mathcal{Q}}(t, \mathbf{q}, \mathbf{u}, \mathbf{p}, \varphi) \doteq D(Pr_1) \cdot F(t, \mathbf{q}, \mathbf{u}, \mathbf{p}, \varphi).$$

**Theorem 3.1** Consider a time interval  $I \subset \mathbb{R}$  and let the maps  $\mathbf{u} : I \mapsto \mathcal{U}$ ,  $(\mathbf{q}, \mathbf{p}) : I \mapsto T^*\mathcal{Q}$  be twice continuously differentiable. Then the following conditions are equivalent:

(i)  $(\mathbf{u}(\cdot), (\mathbf{q}(\cdot), \mathbf{p}(\cdot)))$  is an admissible input-output pair, that is,  $(\mathbf{q}, \mathbf{p})$  verifies

$$\frac{d}{dt}(\mathbf{q}(t), \mathbf{p}(t)) = X_{\mathcal{H}^{\mathbf{u}(t), \dot{\mathbf{u}}(t)}}(\mathbf{q}(t), \mathbf{p}(t)) + \mathbf{F}_{\mathcal{Q}}^{\mathbf{u}(t), \dot{\mathbf{u}}(t)}(\mathbf{q}(t), \mathbf{p}(t)). \quad (3.15)$$

(ii) The path  $(\mathbf{q}(\cdot), \mathbf{p}(\cdot))$  is an integral curve of the control system

$$\frac{d}{dt}(\mathbf{q}(t), \mathbf{p}(t)) = \left[ X_{\mathcal{H}}^{\mathcal{Q}}(t, \mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \varphi) + F_{\mathcal{Q}}(t, \mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \varphi) \right]_{\varphi = \varphi(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t))(\dot{\mathbf{u}}(t))}. \quad (3.16)$$

(iii) There exist selections

$$t \mapsto \varphi(t) \in T_{\mathbf{u}(t)}^*(\mathcal{U}) \quad t \mapsto r(t) \in R_{\mathbf{q}(t), \mathbf{p}(t), \mathbf{u}(t), \varphi(t)}^{\mathcal{Q}}$$

such that, for all  $t \in I$ , one has

$$\frac{d}{dt}(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \varphi(t)) = X_{\mathcal{H}} + \mathbf{F} + r(t). \quad (3.17)$$

The map  $r(\cdot)$  in (3.17) is called the constraint reaction corresponding to the motion  $(\mathbf{q}, \mathbf{p}, \mathbf{u}, \varphi)(\cdot)$ .

**Proof.** Assume that condition **(iii)** holds. In particular, for every  $t \in I$ , one has

$$\dot{\mathbf{u}}(t) = \left( i_{\mathbf{q}(t), \mathbf{u}(t)}^{\mathcal{T}} \right)^{\mathcal{U}} (\mathbf{p}(t), \wp(t)),$$

which implies

$$\wp(t) = \wp(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t)) (\dot{\mathbf{u}}(t)).$$

Therefore

$$\begin{aligned} \frac{d}{dt} (\mathbf{q}(t), \mathbf{p}(t)) &= D(Pr_1) \cdot \frac{d}{dt} (\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \wp(t)) = D(Pr_1) \cdot (X_{\mathcal{H}} + F + r(t)) \\ &= X_{\mathcal{H}}^{\mathcal{Q}}(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \wp(t)) + F_{\mathcal{Q}}(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \wp(t)) \\ &= \left[ X_{\mathcal{H}}^{\mathcal{Q}}(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \wp) + F_{\mathcal{Q}}(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \wp) \right]_{\wp = \wp(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t)) (\dot{\mathbf{u}}(t))}. \end{aligned}$$

Hence **(ii)** holds as well.

Conversely, let us show that **(ii)** implies condition **(iii)**. If  $(\mathbf{q}(\cdot), \mathbf{p}(\cdot))$  is a solution of (3.16), then, setting

$$\wp(t) = \wp(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t)) (\dot{\mathbf{u}}(t)),$$

one has

$$\dot{\mathbf{u}}(t) = \left( i_{\mathbf{q}(t), \mathbf{u}(t)}^{\mathcal{T}} \right)^{\mathcal{U}} (\mathbf{p}(t), \wp(t))$$

which coincides with the equation for the variable  $\mathbf{u}$  in (3.17). Therefore, condition **(iii)** is satisfied provided that we define

$$r(t) \doteq \dot{\wp}(t) - X_{\mathcal{H}}(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \wp(t)) - F(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \wp(t)).$$

In order to prove that conditions **(i)** and **(ii)** are equivalent, it is sufficient to prove that the right-hand sides of the involved equations do coincide. Actually, recalling the definition of  $F_{\mathcal{Q}}^{\mathbf{u}, \mathbf{w}}$  at (3.11), for all  $((\mathbf{q}, \mathbf{p}), (\mathbf{u}, \mathbf{w})) \in T^* \mathcal{Q} \times T^* \mathcal{U}$ , one has

$$\left[ F_{\mathcal{Q}}(t, \mathbf{q}, \mathbf{u}, \mathbf{p}, \wp) \right]_{\wp = \wp(\mathbf{q}, \mathbf{u}, \mathbf{p}) (\mathbf{w})} = F^{\mathbf{u}, \mathbf{w}}(\mathbf{q}, \mathbf{p}).$$

Hence the proof is concluded as soon as one proves the identity

$$\left[ X_{\mathcal{H}}^{\mathcal{Q}}(\mathbf{q}, \mathbf{u}, \mathbf{p}, \wp) \right]_{\wp = \wp(\mathbf{q}, \mathbf{u}, \mathbf{p}) (\mathbf{w})} = X_{\mathcal{H}^{\mathbf{u}, \mathbf{w}}}(\mathbf{q}, \mathbf{p}) \quad (3.18)$$

This will be done in Section 4 —see Lemma 4.1 below— by use of local coordinates.

□

## 4 The control equation in local coordinates

Consider a  $\Lambda$ -adapted coordinate chart  $(q, u)$  defined on an open set  $U$ , and let  $((q, u), (p, w))$  be the corresponding coordinates on the fiber bundle

$$\bigcup_{(\mathbf{q}, \mathbf{u}) \in U} \{(\mathbf{q}, \mathbf{u})\} \times (T_{\mathbf{q}}^* \mathcal{Q} \times T_{\mathbf{u}} \mathcal{U}).$$

Let  $G = (g_{r,s})_{r,s=1,\dots,N+M}$  be the matrix representing the kinetic metric  $\mathbf{g}$ , and let  $G^{-1} = (g^{r,r})_{r,s=1,\dots,N+M}$  denote its inverse. In the following, we consider the sub-matrices

$$G_1 \doteq (g_{i,j}), \quad G_2 \doteq (g_{N+\alpha,N+\beta}), \quad (G^{-1})_2 \doteq (g^{N+\alpha,N+\beta}),$$

$$G_{12} \doteq (g_{i,N+\alpha}), \quad (G^{-1})_{12} \doteq (g^{i,N+\alpha}),$$

with the convention that the Latin indices  $i, j$  run from 1 to  $N$ , while the Greek indices  $\alpha, \beta$  run from 1 to  $M$ . For convenience, we also define

$$A = (a^{i,j}) \doteq (G_1)^{-1}, \quad E = (e_{\alpha,\beta}) \doteq ((G^{-1})_2)^{-1}, \quad K = (k_{N+\alpha}^i) \doteq (G^{-1})_{12} E. \quad (4.1)$$

Let  $\mathbf{u}(\cdot) : I \mapsto \mathcal{U}$  be twice continuously differentiable, and let  $(\mathbf{q}, \mathbf{p}) : I \mapsto T^* \mathcal{Q}$  be a curve such that  $(\mathbf{u}(\cdot), (\mathbf{q}, \mathbf{p})(\cdot))$  is an admissible input-output pair for the control equation of motion. By possibly restricting the size of the interval  $I$ , we can assume that  $(\mathbf{q}(t), \mathbf{u}(t))$  remains inside the domain of the single chart  $(q, u)$  for every  $t \in I$ .

**Theorem 4.1** *Given an admissible input-output pair, the corresponding coordinate maps  $t \mapsto (u(t), (q(t), p^\dagger(t)))$  satisfy the differential equation*

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} Ap \\ -\frac{1}{2} p^\dagger \frac{\partial A}{\partial q} p \end{pmatrix} + \begin{pmatrix} K \dot{u} \\ -p^\dagger \frac{\partial K}{\partial q} \dot{u} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \dot{u}^\dagger \frac{\partial E}{\partial q} \dot{u} \end{pmatrix} + \begin{pmatrix} 0 \\ F_Q^{u(\cdot), \dot{u}(\cdot)} \end{pmatrix}. \quad (4.2)$$

In addition, recalling (3.10)-(3.11),

$$F_Q^{u(\cdot), \dot{u}(\cdot)} \doteq (F_1^{u(\cdot), \dot{u}(\cdot)}, \dots, F_N^{u(\cdot), \dot{u}(\cdot)}). \quad (4.3)$$

For convenience, in (4.2) we write all vectors as column vectors, while the superscript  $^\dagger$  denotes transposition. Componentwise, this means:

$$\begin{cases} \dot{q}^i = a^{i,j} p_j & + & k_{N+\alpha}^i \dot{u}^\alpha, \\ \dot{p}_i = -\frac{1}{2} \frac{\partial a^{\ell,j}}{\partial q^i} p_\ell p_j & - & \frac{\partial k_\alpha^j}{\partial q^i} p_j \dot{u}^\alpha + \frac{1}{2} \frac{\partial e_{\alpha,\beta}}{\partial q^i} \dot{u}^\alpha \dot{u}^\beta + F_i^{u(\cdot), \dot{u}(\cdot)}. \end{cases} \quad (4.4)$$

(where  $\ell$  runs from 1 to  $N$ ).



**Proof.** Let  $\mathcal{H}^{u,w}$  be the local coordinate representation of the Hamiltonian  $\mathcal{H}^{\mathbf{u},\mathbf{w}}$ . Then the map  $(q, p)(\cdot)$  satisfies the system

$$\begin{cases} \dot{q}^i(t) = \frac{\partial \mathcal{H}^{u(t), \dot{u}(t)}}{\partial p_i}(q(t), p(t)), \\ \dot{p}_i(t) = -\frac{\partial \mathcal{H}^{u(t), \dot{u}(t)}}{\partial q^i}(q(t), p(t)) + F_i^{u(t), \dot{u}(t)}(t, q(t), p(t)). \end{cases} \quad (4.5)$$

Using the expression of the kinetic energy

$$\mathcal{T}^{u,w} = \frac{1}{2} \dot{q}^\dagger G_1 \dot{q} + \dot{q}^\dagger G_{12} w + \frac{1}{2} w^\dagger G_2 w$$

in terms of the given coordinates, we obtain

$$\mathcal{H}^{u,w}(q, p) = \mathcal{H}_2^{u,w}(q, p) + \mathcal{H}_1^{u,w}(q, p) + \mathcal{H}_0^{u,w}(q, p), \quad (4.6)$$

where

$$\begin{aligned} \mathcal{H}_2^{u,w}(q, p) &= \frac{1}{2} p^\dagger (G_1)^{-1} p, & \mathcal{H}_1^{u,w}(q, p) &= -p^\dagger (G_1)^{-1} G_{12} w, \\ \mathcal{H}_0^{u,w}(q, p) &= \frac{1}{2} w^\dagger G_{12}^\dagger (G_1)^{-1} G_{12} w - \frac{1}{2} w^\dagger G_2 w. \end{aligned}$$

The theorem can thus be proved by checking that

$$\mathcal{H}^{u,w}(q, u) = \frac{1}{2} p^\dagger A p + p^\dagger K w - \frac{1}{2} w^\dagger E w. \quad (4.7)$$

From the identity

$$G_1 (G^{-1})_{12} + G_{12} (G^{-1})_2 = 0$$

one obtains

$$0 = G_1^{-1} G_1 (G^{-1})_{12} E + G_1^{-1} G_{12} (G^{-1})_2 E,$$

which implies

$$(K =) \quad (G^{-1})_{12} E = -(G_1)^{-1} G_{12} = -A G_{12}. \quad (4.8)$$

Moreover, from (4.8) and the identity

$$\mathbf{I}_M = G_{12}^\dagger (G^{-1})_{12} + G_2 (G^{-1})_2,$$

one gets

$$E = G_{12}^\dagger (G^{-1})_{12} E + G_2 (G^{-1})_2 E = -G_{12}^\dagger A G_{12} + G_2. \quad (4.9)$$

Together, (4.8) and (4.9) yield (4.7), concluding the proof.  $\square$

We conclude this section by proving the following lemma, which was used in the proof of Theorem 3.1.

**Lemma 4.1** *For all  $((\mathbf{q}, \mathbf{p}), (\mathbf{u}, \mathbf{w})) \in T^* \mathcal{Q} \times T\mathcal{U}$  one has*

$$\left[ X_{\mathcal{H}}^{\mathcal{Q}}(\mathbf{q}, \mathbf{u}, \mathbf{p}, \wp) \right]_{\wp = \wp(\mathbf{q}, \mathbf{u}, \mathbf{p})(\mathbf{w})} = X_{\mathcal{H}^{\mathbf{u}, \mathbf{w}}}(\mathbf{q}, \mathbf{p}) \quad (4.10)$$

**Proof.** Since the thesis is local one can use coordinates to prove it. Consider a  $\Lambda$ -adapted coordinate chart  $(q, u)$  and let  $((q, u), (p, w))$   $((q, u), (p, \wp))$  be the corresponding coordinates on the fiber bundles  $T^*\mathcal{Q} \times T\mathcal{U}$  and  $T^*\mathcal{Q} \times T^*\mathcal{U}$ , respectively. Let  $(G^{-1})_{12}, A, E, K$  have the same meaning as before. Moreover, let us set  $((G^{-1})_1 \doteq (g^{i,j})_{i,j=1,\dots,N})$ . We have to prove that

$$\frac{\partial \mathcal{H}^{u,w}}{\partial p} = \frac{\partial \mathcal{H}}{\partial p} \Big|_{\wp = \wp(q,u,p)(w)} \quad (4.11)$$

and

$$\frac{\partial \mathcal{H}^{u,w}}{\partial q} = \frac{\partial \mathcal{H}}{\partial q} \Big|_{\wp = \wp(q,u,p)(w)}, \quad (4.12)$$

where

$$\wp(q,u,p)(w) \doteq w^\dagger E + p^\dagger K.$$

In fact, by (4.6) one has

$$\frac{\partial \mathcal{H}^{u,w}}{\partial p} = p^\dagger A + K w. \quad (4.13)$$

On the other hand, one can easily check that

$$\frac{\partial \mathcal{H}}{\partial p} \Big|_{\wp = \wp(q,u,p)(w)} = p^\dagger (G^{-1})_1 + (G^{-1})_{12} (w^\dagger E + p^\dagger K) = p^\dagger A + K w, \quad (4.14)$$

so (4.11) is proved.

To prove (4.12), let us consider the well-known identities

$$\frac{\partial \mathcal{H}}{\partial q}(q, u, p, \wp) = - \frac{\partial \mathcal{T}}{\partial q}(q, u, v, w) \Big|_{(v,w) = \left( \left( i_{\mathbf{q},\mathbf{u}}^{\mathcal{H}} \right)^{\mathcal{Q}}_{(p,\wp)}, \left( i_{\mathbf{q},\mathbf{u}}^{\mathcal{H}} \right)^{\mathcal{U}}_{(p,\wp)} \right)}$$

$$\frac{\partial \mathcal{H}^{u,w}}{\partial q}(q, p) = - \frac{\partial \mathcal{T}^{u,w}}{\partial q}(q, v) \Big|_{v = \frac{\partial \mathcal{H}^{u,w}}{\partial p}(q, p)}$$

Observe that the map  $p \mapsto \frac{\partial \mathcal{H}^{u,w}}{\partial p}(q, p)$  is the inverse of the map  $v \mapsto \frac{\partial \mathcal{T}^{u,w}}{\partial v}(q, v)$ . Hence, letting

$$\begin{aligned} \check{v} &\doteq \left( i_{\mathbf{q},\mathbf{u}}^{\mathcal{H}} \right)^{\mathcal{Q}} \left( p, \left( i_{\mathbf{q},\mathbf{u}}^{\mathcal{T}} \right)^{\mathcal{U}} \left( \frac{\partial \mathcal{H}^{u,w}}{\partial p}, w \right) \right) \\ \check{w} &\doteq \left( i_{\mathbf{q},\mathbf{u}}^{\mathcal{H}} \right)^{\mathcal{U}} \left( p, \left( i_{\mathbf{q},\mathbf{u}}^{\mathcal{T}} \right)^{\mathcal{U}} \left( \frac{\partial \mathcal{H}^{u,w}}{\partial p}, w \right) \right), \end{aligned}$$

one obtains

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial q}(q, u, p, \wp) \Big|_{\wp = \wp(q,u,p,w)} &= \frac{\partial \mathcal{H}}{\partial q}(q, u, p, \wp) \Big|_{\wp = \left( i_{\mathbf{q},\mathbf{u}}^{\mathcal{T}} \right)^{\mathcal{U}} \left( \frac{\partial \mathcal{H}^{u,w}}{\partial p}, w \right)} = \\ - \frac{\partial \mathcal{T}}{\partial q}(q, u, v, w) \Big|_{(v,w) = (\check{v}, \check{w})} &= - \frac{\partial \mathcal{T}}{\partial q}(q, u, v, w) \Big|_{v = \frac{\partial \mathcal{H}^{u,w}}{\partial p}(q, p)} = \frac{\partial \mathcal{H}^{u,w}}{\partial q}(q, p), \end{aligned}$$

so (4.12) is proved as well.  $\square$

**Remark 4.1** As a matter of fact, one could show that Lemma 4.1 is valid under the weaker assumption that  $\mathcal{T}$  is strictly convex in the velocity  $(\mathbf{v}, \mathbf{w})$  — see [25].

## 5 The Riemannian structure and the input-output map

The presence of the derivative  $\dot{\mathbf{u}}$  in the dynamic equations (4.2) depends on the Riemannian metric  $g$  defining the kinetic energy and on the foliation  $\Lambda$  at (3.3). In this section we review the main results in this direction. To simplify the discussion, throughout this section we shall assume that the additional forces  $\mathbf{F}$  vanish identically, so that in (4.3) one has

$$F_Q^{u(\cdot), \dot{u}(\cdot)} \equiv 0.$$

The following definitions were introduced in [8].

**Definition 5.1** *A local,  $\Lambda$ -adapted, system of coordinates  $(q, u)$  on  $\mathcal{Q} \times \mathcal{U}$  is called  $N$ -fit for hyperimpulses if, for every differentiable control function  $\mathbf{u}(\cdot)$ , the right-hand side*

*of the corresponding equation of motion (4.4) does not contains any quadratic term in the variable  $\dot{u}$ .*

*A local,  $\Lambda$ -adapted, system of coordinates  $(q, u)$  on  $\mathcal{Q} \times \mathcal{U}$  is called strongly  $N$ -fit for hyperimpulses if, for every differentiable control function  $\mathbf{u}(\cdot)$ , right-hand side of (4.4) is independent of the variable  $\dot{u}$ .*

*Moreover, we shall call generic any local,  $\Lambda$ -adapted, system of coordinates  $(q, u)$  which is not  $N$ -fit for hyperimpulses.*

**Remark 5.1** The denomination “ $N$ -fit for hyperimpulses” for a system of coordinates  $(q, u)$  refers to the fact that, if the dependence on  $\dot{u}$  is only linear, one can then construct solutions  $(q(\cdot), p(\cdot))$  also for discontinuous controls  $u(\cdot)$ . In general, a jump in  $u(\cdot)$  will produce a discontinuity in both components  $q(\cdot)$  and  $p(\cdot)$ . For this reason we call it a *hyperimpulse*, as opposite to *impulse*, which can cause a discontinuity in the component  $p(\cdot)$  only.

A first characterization of  $N$ -fit coordinates was derived in [8]. It is important to observe that the property of being  $N$ -fit depends only on the metric  $g$  and on the foliation  $\Lambda$ , not on particular the system of  $\Lambda$ -adapted coordinates. This leads to

**Definition 5.2** [23] *The foliation  $\Lambda$  is called  $N$ -fit for hyperimpulses if there exists an atlas of  $\Lambda$ -adapted charts that are also  $N$ -fit for hyperimpulses. In this case, all  $\Lambda$ -adapted charts are then  $N$ -fit for hyperimpulses.*

*The foliation  $\Lambda$  is called strongly  $N$ -fit for hyper-impulses if there exists an atlas of  $\Lambda$ -adapted charts which are strongly  $N$ -fit for hyper-impulses.*

*Moreover, the foliation  $\Lambda$  will be called generic if it is not  $N$ -fit for hyper-impulses.*

The paper [23] established the connection between the  $N$ -fitness of the foliation  $\Lambda$  and the bundle-like property of the metric, introduced in [27, 28]. We recall here the main definitions and results.

**Definition 5.3** *The metric  $\mathbf{g}$  is bundle-like with respect to the foliation  $\Lambda$  if, for every  $\Lambda$ -adapted chart, it has a local representation of the form*

$$\sum_{i,j=1}^N g_{i,j}(q,u) \omega^i \otimes \omega^j + \sum_{\alpha,\beta=1}^M g_{N+\alpha,N+\beta}(q,u) dc^\alpha \otimes dc^\beta,$$

where  $\omega^1, \dots, \omega^N$  are linearly independent 1-forms such that, for each  $(\mathbf{q}, \mathbf{u}) \in \mathcal{Q} \times \mathcal{U}$  in the domain of the chart, one has

- (i)  $(\omega^1(\mathbf{q}, \mathbf{u}), \dots, \omega^N(\mathbf{q}, \mathbf{u}), dc^1(\mathbf{u}), \dots, dc^M(\mathbf{u}))$  is a basis of the cotangent space  $T_{\mathbf{q}}^* \mathcal{Q} \times T_{\mathbf{u}}^* \mathcal{U}$ ;
- (ii)  $\langle \omega^i(\mathbf{q}, \mathbf{u}), Y \rangle = 0$ , for every  $Y \in \Delta_{(\mathbf{q}, \mathbf{u})}^\perp$ .

We recall that  $\Delta_{(\mathbf{q}, \mathbf{u})}^\perp$  is the orthogonal bundle, defined at (3.4). If  $g$  is bundle-like with respect to the foliation  $\Lambda$ , the latter is also called a *Riemannian foliation*, because in this case a Riemannian structure can be well defined also on the quotient space. In order to state the next theorem, we recall the notion of *completely integrable* distribution. (Here, by *distribution* we mean a map that maps

**Definition 5.4** *Let  $\mathcal{Y}$  be a manifold of dimension  $d$ , and let  $\Gamma$  be a distribution on  $\mathcal{Y}$  of dimension  $N \leq d$ . (I.e., for every  $y \in M$ ,  $\Gamma(y)$  is a subspace of dimension  $N$  of  $T_y \mathcal{Y}$ .) We say that the distribution  $\Gamma$  is completely integrable if, for every  $\mathbf{y} \in \mathcal{Y}$ , there exists a neighborhood  $U$  of  $\mathbf{y}$  and a local system of coordinates  $(x, z) = (x^1, \dots, x^N, z^{N+1}, \dots, z^d)$  such that at each point  $\mathbf{y} \in U$  one has*

$$\Gamma_{\mathbf{y}} = \text{span} \left\{ \frac{\partial}{\partial x_i} \quad i = 1, \dots, N \right\}.$$

**Theorem 5.1** *On the product space  $\mathcal{Y} = \mathcal{Q} \times \mathcal{U}$ , consider the natural foliation  $\Lambda$  as in (3.3). The following statements are equivalent:*

- i) *The foliation  $\Lambda$  is  $N$ -fit for hyper-impulses.*
- ii) *The metric  $\mathbf{g}$  is bundle-like with respect to the foliation  $\Lambda$ , i.e., the foliation  $\Lambda$  is Riemannian.*
- iii) *For any  $\mathbf{u}, \bar{\mathbf{u}} \in \mathcal{U}$  the map  $d_{\mathbf{u}, \bar{\mathbf{u}}}(\cdot) : \mathcal{Q} \mapsto \mathbb{R}$  defined by*

$$d_{\mathbf{u}}(\mathbf{q}) \doteq \text{dist} \left( (\mathbf{q}, \mathbf{u}), \mathcal{Q} \times \{\bar{\mathbf{u}}\} \right)$$

*is constant. In other words, leaves remain at the same distance from each other.*

- iv) *If  $t \mapsto (\mathbf{q}(t), \mathbf{u}(t))$  is any geodesic curve with respect to the metric  $\mathbf{g}$ , and if  $(\dot{\mathbf{q}}(\tau), \dot{\mathbf{u}}(\tau)) \in \Delta_{(\mathbf{q}(\tau), \mathbf{u}(\tau))}^\perp$  at some time  $\tau$ , then  $(\dot{\mathbf{q}}(t), \dot{\mathbf{u}}(t)) \in \Delta_{(\mathbf{q}(t), \mathbf{u}(t))}^\perp$  for all  $t$ . In other words, if a geodesic crosses perpendicularly one of the leaves, then it crosses perpendicularly also every other leaf which it meets.*

v) If  $(q, u)$  is a  $\Lambda$ -adapted system of coordinates, then

$$\frac{\partial g^{N+\alpha, N+\beta}}{\partial q_i} = 0 \quad i = 1, \dots, N, \quad \alpha, \beta = 1, \dots, M, \quad (5.1)$$

where  $G^{-1} = (g^{r,s})$  denotes the inverse of the matrix  $G = (g_{r,s})$  representing the metric  $\mathbf{g}$  in the coordinates  $(q, u)$ .

Indeed, the equivalence of i) and ii) is a trivial consequence of the definitions of bundle-like metric and of  $N$ -fit system of coordinates. The equivalence of ii), iii), and iv), is a classical result on bundle-like metrics [27]. Moreover, by (4.4), the foliation is fit for jumps if and only if  $\partial e_{\alpha,\beta}/\partial q^i \equiv 0$ . Recalling that the matrix  $(e_{\alpha,\beta})$  is the inverse of  $(G^{-1})_2 = (g^{N+\alpha, N+\beta})$ , we conclude that i) is equivalent to v).

**Theorem 5.2** *The following statements are equivalent:*

- i) *The foliation  $\Lambda$  is strongly  $N$ -fit for hyperimpulses .*
- ii) *The foliation  $\Lambda$  is  $N$ -fit for hyperimpulses and the orthogonal bundle  $\Delta_{(\mathbf{q}, \mathbf{u})}^\perp$  in (3.4) is integrable.*
- iii) *There is an atlas such that, for every chart  $(q, u)$ , one has*

$$\frac{\partial g^{N+\alpha, N+\beta}}{\partial q_i} = 0, \quad g^{i, N+\alpha} = 0 \quad \text{for all } i = 1, \dots, N, \quad \alpha, \beta = 1, \dots, M.$$

Indeed the equivalence of i) and ii), formulated in terms of Riemannian foliations, was proved in [27]. The equivalence between i) and iii) follows from (4.4). See again [27] and [23] for details.

## 6 The quadratic term in the control equation and the orthogonal curvature of the foliation $\Lambda$

As we have seen in the previous section, the  $N$ -fitness for hyperimpulses of a coordinate system  $(q, u)$  can be characterized in terms of geodesics. Indeed, the quadratic terms in the control equation of motion (4.4) are identically zero if and only if any geodesic which crosses perpendicularly one leaf of the foliation  $\Lambda$  also has perpendicular intersection with every other leaf it meets.

In the general case, however, the quadratic terms in (4.4) do not vanish. We wish to give here a geometric interpretation of these terms. This will again be achieved by looking at geodesics whose tangent vector initially lies in the orthogonal distribution  $\Delta^\perp$ .

### 6.1 $\mathcal{U}$ -orthonormal coordinates

We shall make an essential use of a Proposition 6.1 below, which establishes the existence of a special kind of  $\Lambda$ -adapted charts. To state it, let us set  $(x^1, \dots, x^{N+M}) \doteq (q^1, \dots, q^N, u^1, \dots, u^M)$ ,

and, for every  $h, k, r, s = 1, \dots, N + M$ , let us consider the functions

$$\Gamma_{h,r,s} \doteq \frac{1}{2} \left( \frac{\partial g_{h,r}}{\partial x^s} + \frac{\partial g_{h,s}}{\partial x^r} - \frac{\partial g_{r,s}}{\partial x^h} \right) \quad \Gamma_{r,s}^k \doteq g^{kh} \left( \frac{\partial g^{h,r}}{\partial x^s} + \frac{\partial g^{h,s}}{\partial x^r} - \frac{\partial g^{r,s}}{\partial x^h} \right)$$

The  $\Gamma_{r,s}^k$  are the well-known Christoffel symbols.

**Proposition 6.1** *Consider a point  $(\bar{\mathbf{q}}, \bar{\mathbf{u}}) \in \mathcal{Q} \times \mathcal{U}$  and an orthonormal basis  $\{J_1, \dots, J_M\}$  of  $\Delta^\perp(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ . Then there exist  $\Lambda$ -adapted coordinates  $(q, u)$ , defined on a neighborhood of  $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ , such that calling  $G = (g_{r,s})_{r,s=1,\dots,N+M}$  the corresponding kinetic matrix, one has*

- (i) *the point  $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$  has coordinates  $(0, 0)$ ;*
- (ii)  *$g_{r,s}(0, 0) = g^{r,s}(0, 0) = \delta_{r,s}$  (the Kronecker symbol) for all  $r, s = 1, \dots, N + M$ ;*
- (iii) *for every  $w = (w_1, \dots, w_M) \in \mathbb{R}^M$ , the geodesic  $(q, u)_w(\cdot)$  issuing from  $(\mathbf{q}, \mathbf{u})$  with velocity equal to  $w_1 J_1 + \dots + w_M J_M$  has local representation  $(q, u)_w(t) = (0, \dots, 0, tw_1, \dots, tw_M)$ .*

Moreover, for all indices  $i = 1, \dots, N$  and  $\alpha, \beta, \gamma = 1, \dots, M$ , we have

$$\Gamma_{i,N+\alpha,N+\beta}(0, 0) = \Gamma_{N+\alpha,N+\beta}^i(0, 0) = 0, \quad \Gamma_{N+\gamma,N+\alpha,N+\beta}(0, 0) = \Gamma_{N+\alpha,N+\beta}^{N+\gamma}(0, 0) = 0 \quad (6.1)$$

In turn, this implies

$$\frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0, 0) = \frac{\partial g^{i,N+\beta}}{\partial u^\gamma}(0, 0) + \frac{\partial g^{i,N+\gamma}}{\partial u^\beta}(0, 0), \quad (6.2)$$

$$\frac{\partial g_{N+\alpha,N+\beta}}{\partial u^\gamma}(0, 0) = \frac{\partial g^{N+\alpha,N+\beta}}{\partial u^\gamma}(0, 0) = 0. \quad (6.3)$$

A chart with the above properties will be called  $\mathcal{U}$ -orthonormal at  $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ .

**Proof.** We start by considering  $\Lambda$ -adapted coordinates  $(\hat{q}, \hat{u})$ , defined on a neighborhood of the point  $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ , such that at the point  $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$  one has  $(\hat{q}, \hat{u}) = (0, 0)$  and

$$\frac{\partial}{\partial \hat{u}^\alpha} = J_\alpha, \quad \alpha = 1, \dots, M,$$

while

$$\left\{ \frac{\partial}{\partial \hat{q}^i}; \quad i = 1, \dots, N \right\}$$

is an orthonormal basis of the tangent space  $T\mathcal{Q}$  at  $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ , w.r.t. the metric  $\mathbf{g}$ .

To achieve the further property (iii), we need to modify these coordinates, using the exponential map. In the following, given a tangent vector  $\mathbf{V} \in T_{(\bar{\mathbf{q}}, \bar{\mathbf{u}})}(\mathcal{Q} \times \mathcal{U})$ , we denote by  $\tau \mapsto \gamma_{\mathbf{V}}(\tau)$  the geodesic curve starting from  $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$  with velocity  $\mathbf{V}$ . In other words,

$$\gamma_{\mathbf{V}}(0) = (\bar{\mathbf{q}}, \bar{\mathbf{u}}), \quad \frac{d\gamma_{\mathbf{V}}}{d\tau}(0) = \mathbf{V}.$$

The exponential map is then defined by setting

$$\text{Exp}_{(\bar{\mathbf{q}}, \bar{\mathbf{u}})}(\mathbf{V}) \doteq \gamma_{\mathbf{V}}(1).$$

This is well defined for all vectors  $\mathbf{V}$  in a neighborhood of the origin.

Denote by  $(\hat{q}, \hat{u})(\mathbf{q}, \mathbf{u})$  the coordinates of a point  $(\mathbf{q}, \mathbf{u})$  via the chart  $(\hat{q}, \hat{u})$ . We now define  $(q, u)$  to be the new coordinates of a point  $(\mathbf{q}, \mathbf{u})$  provided that

$$(\hat{q}, \hat{u})(\mathbf{q}, \mathbf{u}) = (\hat{q}, \hat{u}) \left( \text{Exp}_{(\bar{\mathbf{q}}, \bar{\mathbf{u}})} \left( \sum_{\alpha=1}^M u^\alpha J_\alpha \right) \right) + (q^1, \dots, q^N, 0, \dots, 0). \quad (6.4)$$

Notice that this is well defined, for all  $(\mathbf{q}, \mathbf{u})$  in a neighborhood of  $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ . Indeed, the map  $\rho: \mathbb{R}^{N+M} \mapsto \mathbb{R}^{N+M}$  defined by

$$\rho(q^1, \dots, q^N, u^1, \dots, u^M) = (\hat{q}, \hat{u}) \left( \text{Exp}_{(\bar{\mathbf{q}}, \bar{\mathbf{u}})} \left( \sum_{\alpha=1}^M u^\alpha J_\alpha \right) \right) + (q^1, \dots, q^N, 0, \dots, 0). \quad (6.5)$$

maps the origin into itself. Moreover, by the properties of the chart  $(\hat{q}, \hat{u})$ , the Jacobian matrix  $\partial \rho / \partial (q^1, \dots, q^N, u^1, \dots, u^M)$  at the origin coincides with the identity matrix. This already establishes the properties (i)-(ii).

By construction,  $(q, u)(\mathbf{q}, \mathbf{u}) = (0, \dots, 0, u^1, \dots, u^M)$  if and only if

$$(\mathbf{q}, \mathbf{u}) = \text{Exp}_{(\bar{\mathbf{q}}, \bar{\mathbf{u}})} \left( \sum_{\alpha=1}^M u^\alpha J_\alpha \right).$$

This establishes (iii).

In order to prove (6.2)-(6.3), we observe that the geodesic curves correspond to solutions of the second order equations

$$\begin{cases} \ddot{q}^i = \Gamma_{j,\ell}^i \dot{q}^j \dot{q}^\ell + 2\Gamma_{j,N+\alpha}^i \dot{q}^j \dot{u}^\alpha + \Gamma_{N+\alpha,N+\beta}^i \dot{u}^\alpha \dot{u}^\beta & i = 1, \dots, N \\ \ddot{u}^\gamma = \Gamma_{j,\ell}^\gamma \dot{q}^j \dot{q}^\ell + 2\Gamma_{j,N+\alpha}^\gamma \dot{q}^j \dot{u}^\alpha + \Gamma_{N+\alpha,N+\beta}^\gamma \dot{u}^\alpha \dot{u}^\beta & \gamma = 1, \dots, M \end{cases} \quad (6.6)$$

By the previous construction, for any given  $w \in \mathbb{R}^M$  the solution of (6.6) with initial data

$$(q, u, \dot{q}, \dot{u})(0) = (0, 0, 0, w) \quad (6.7)$$

satisfies

$$q(t) \equiv 0, \quad u(t) = t w. \quad (6.8)$$

By (6.6) we obtain

$$0 = \ddot{q}^i(0) = \Gamma_{N+\alpha,N+\beta}^i(0, 0) w^\alpha w^\beta$$

$$0 = \ddot{u}^\gamma(0) = \Gamma_{N+\alpha,N+\beta}^\gamma(0, 0) w^\alpha w^\beta$$

Since these equalities hold for all initial data  $w \in \mathbb{R}^M$  in (6.7), this and property (ii) prove (6.1).

Now, by

$$0 = \Gamma_{i,N+\alpha,N+\beta}(0, 0) = \frac{\partial g_{i,N+\beta}}{\partial u^\gamma}(0, 0) + \frac{\partial g_{i,N+\gamma}}{\partial u^\beta}(0, 0) - \frac{\partial g_{N+\beta,N+\gamma}}{\partial q^i}(0, 0)$$

we obtain 6.2, since, by property (ii), one has

$$\frac{\partial g_{i,N+\beta}}{\partial u^\gamma}(0,0) = -\frac{\partial g^{i,N+\beta}}{\partial u^\gamma}(0,0), \quad \frac{\partial g_{N+\beta,N+\gamma}}{\partial q^i}(0,0) = -\frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0,0)$$

for all  $i = 1, \dots, N$  and  $\alpha, \beta = 1, \dots, M$ .

Moreover, for every  $\alpha, \beta, \gamma = 1, \dots, M$ , one has

$$\frac{\partial g_{N+\alpha,N+\beta}}{\partial u^\gamma} = \Gamma_{N+\alpha,N+\gamma,N+\beta} + \Gamma_{N+\beta,N+\gamma,N+\alpha} = 0,$$

so that, in view of property (ii), (6.3) is proved as well.  $\square$

**Remark 6.1** As in (4.1), we can now define  $(e_{\alpha,\beta})_{\alpha,\beta=1,\dots,M}$  as the inverse of the sub-matrix  $(g^{N+\alpha,N+\beta})_{\alpha,\beta=1,\dots,M}$ . Then, since at  $(q, u) = (0, 0)$  we have  $g_{r,s} = g^{r,s} = \delta_{r,s}$ , it follows

$$\frac{\partial e_{\alpha,\beta}}{\partial q^i}(0,0) = \frac{\partial g_{N+\alpha,N+\beta}}{\partial q^i}(0,0) \quad \text{for all } i = 1, \dots, N, \quad \alpha, \beta = 1, \dots, M. \quad (6.9)$$

## 6.2 The orthogonal curvature of the foliation

For any  $(q, u)$  in the range of a  $\Lambda$ -adapted chart, consider the quantity

$$\frac{\partial e_{\alpha,\beta}}{\partial q^i} dc^\alpha \otimes dc^\beta \otimes dq^i \quad (6.10)$$

**Lemma 6.1** *The function in (6.10) is intrinsically defined with respect to the foliation  $\Lambda$ . This means that if  $(\tilde{q}, \tilde{u})$  is a  $\Lambda$ -adapted chart then*

$$\frac{\partial \tilde{e}_{\alpha,\beta}}{\partial \tilde{q}^i} = \frac{\partial u^\gamma}{\partial \tilde{u}^\alpha} \frac{\partial u^\delta}{\partial \tilde{u}^\beta} \frac{\partial q^j}{\partial \tilde{q}^i} \frac{\partial e_{\gamma,\delta}}{\partial q^j}. \quad (6.11)$$

**Proof.** Since  $(q, u)$  and  $(\tilde{q}, \tilde{u})$  are  $\Lambda$ -adapted, the coordinate transformation  $(q, u) \mapsto (\tilde{q}, \tilde{u})$  satisfies  $\frac{\partial \tilde{u}}{\partial q} = 0$ . Therefore,

$$\tilde{g}^{N+\alpha,N+\beta} = \frac{\partial \tilde{u}^\alpha}{\partial u^\gamma} \frac{\partial \tilde{u}^\beta}{\partial u^\delta} g^{N+\gamma,N+\delta}.$$

By inverting the matrices on both sides of the above identity one obtains

$$\tilde{e}_{\alpha,\beta} = \frac{\partial u^\gamma}{\partial \tilde{u}^\alpha} \frac{\partial u^\delta}{\partial \tilde{u}^\beta} e_{\gamma,\delta},$$

which implies (6.11), because  $u = u(\tilde{u})$  is independent of  $\tilde{q}$ .  $\square$



Although the quantity in (6.10) is not a tensor in the strict sense of the word, by (6.11) it still transforms like a tensor w.r.t. to changes of  $\Lambda$ -adapted coordinates. Hence, it is intrinsically defined in terms of the foliation. By a slight abuse of language, we thus define (6.10) as the *orthogonal curvature tensor of the foliation*  $\Lambda$ .

According to Theorem 5.1, the foliation  $\Lambda$  is  $N$ -fit for hyperimpulses if and only if the corresponding orthogonal curvature is identically equal to zero. We now give a geometric construction which clarifies the meaning of the coefficients  $\partial e_{\alpha,\beta}/\partial q^i$  in (6.10), in the general case (see Figure 2).

Fix any point  $(\mathbf{q}, \mathbf{u}) \in \mathcal{Q} \times \mathcal{U}$  and consider any non-zero vector  $\mathbf{V} \in \Delta_{(\mathbf{q}, \mathbf{u})}^\perp$ . Construct the geodesic curve that originates at  $(\mathbf{q}, \mathbf{u})$  with speed  $\mathbf{V}$ , namely

$$s \mapsto \gamma_{\mathbf{V}}(s) \doteq \text{Exp}_{(\mathbf{q}, \mathbf{u})}(s\mathbf{V}). \quad (6.12)$$

Next, for each  $s \neq 0$ , consider the orthogonal space  $\Delta_{(\mathbf{q}_s, \mathbf{u}_s)}^\perp$  at the point  $(\mathbf{q}_s, \mathbf{u}_s) = \gamma_{\mathbf{V}}(s)$ . Assuming that  $s$  is sufficiently small, a transversality argument yields the existence of a unique vector  $\mathbf{W} \in \Delta_{(\mathbf{q}_s, \mathbf{u}_s)}^\perp$  such that

$$\text{Exp}_{(\mathbf{q}_s, \mathbf{p}_s)}\mathbf{W} = (\hat{\mathbf{q}}_s, \mathbf{u}) \in \mathcal{Q} \times \{\mathbf{u}\}. \quad (6.13)$$

In other words, we are moving back to a point  $(\hat{\mathbf{q}}_s, \mathbf{u})$  on the original leaf  $\mathcal{Q} \times \{\mathbf{u}\}$ , following a second geodesic curve. In general,  $\hat{\mathbf{q}}_s \neq \mathbf{q}$ . We claim that, setting  $\sigma \doteq s^2$ , the map

$$\sigma \mapsto (\hat{\mathbf{q}}_{\sqrt{\sigma}}, \mathbf{u})$$

defines a unique tangent vector  $\mathbf{Z}(\mathbf{V}) \in T_{(\mathbf{q}, \mathbf{u})}$ . Moreover, the map  $\mathbf{V} \mapsto \mathbf{Z}(\mathbf{V})$  is a homogeneous quadratic map from  $\Delta_{(\mathbf{q}, \mathbf{u})}^\perp$  into the tangent space  $T_{(\mathbf{q}, \mathbf{u})}\mathcal{Q} \subset T_{(\mathbf{q}, \mathbf{u})}(\mathcal{Q} \times \mathcal{U})$ . In turn, this determines a unique symmetric bilinear mapping  $B : \Delta_{(\mathbf{q}, \mathbf{u})}^\perp \otimes \Delta_{(\mathbf{q}, \mathbf{u})}^\perp \mapsto T_{(\mathbf{q}, \mathbf{u})}\mathcal{Q}$  such that  $B(\mathbf{V}, \mathbf{V}) = \mathbf{Z}(\mathbf{V})$ , namely

$$B(\mathbf{V}_1, \mathbf{V}_2) \doteq \frac{1}{4}\mathbf{Z}(\mathbf{V}_1 + \mathbf{V}_2) - \frac{1}{4}\mathbf{Z}(\mathbf{V}_1 - \mathbf{V}_2). \quad (6.14)$$

The relation between the bilinear mapping (6.14) and the curvature tensor (6.10) can be best analyzed by using coordinates. Consider an orthonormal basis  $(J_1, \dots, J_M)$  of  $\Delta_{(\mathbf{q}, \mathbf{u})}^\perp$ , together with local  $\mathcal{U}$ -orthonormal coordinates  $(q, u)$ , constructed as in Proposition 6.1. If  $\mathbf{V} = w_1 J_1 + \dots + w_M J_M$ , then by construction the point  $(\mathbf{q}_s, \mathbf{u}_s)$  has coordinates  $(0, sw) = (0, \dots, 0, sw_1, \dots, sw_M)$ . Let  $(\hat{q}_w(s), 0)$  be the coordinates of the point  $(\hat{\mathbf{q}}_s, \mathbf{u})$ , constructed as in (6.13). We now have:

**Theorem 6.1** *The curve  $s \mapsto q_w(s) \in \mathbb{R}^N$  is continuous and satisfies*

$$\lim_{s \rightarrow 0} \frac{\hat{q}_w^i(s)}{s^2} = \frac{1}{2} \sum_{\alpha, \beta=1}^M \frac{\partial e_{\alpha, \beta}}{\partial q^i} w^\alpha w^\beta \quad i = 1, \dots, N. \quad (6.15)$$

**Proof.** It is understood that the coefficients  $\partial e_{\alpha, \beta}/\partial q^i$  in (6.15) are computed at  $(q, u) = (0, 0)$ , corresponding to the point  $(\mathbf{q}, \mathbf{u})$ . In view of (6.9), it suffices to prove that

$$\lim_{s \rightarrow 0} \frac{\hat{q}_w^i(s)}{s^2} = \frac{1}{2} \sum_{\alpha, \beta=1}^M \frac{\partial g_{N+\alpha, N+\beta}}{\partial q^i} w^\alpha w^\beta. \quad (6.16)$$

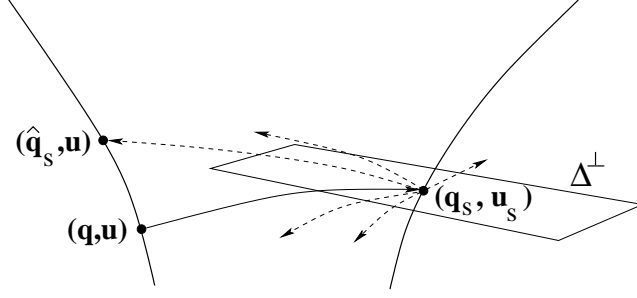


Figure 2: The geodesics involved in the computation of the orthogonal curvature of  $\Lambda$ .

In coordinates, the geodesic  $\sigma \mapsto \gamma_{\mathbf{W}}(\sigma) = \text{Exp}_{(\mathbf{q}_s, \mathbf{u}_s)}(\sigma \mathbf{W})$  is given by a map  $\sigma \mapsto (\hat{q}(\sigma), \hat{u}(\sigma))$  which, for suitable adjoint variables  $p = (p_1, \dots, p_N)$ ,  $\pi = (\pi_1, \dots, \pi_M)$ , satisfies the Hamiltonian system

$$\begin{cases} \dot{q}^i = g^{i,j} p_j + g^{i,N+\beta} \pi_\beta \\ \dot{u}^\alpha = g^{N+\alpha,j} p_j + g^{N+\alpha,N+\beta} \pi_\beta \\ \dot{p}_i = -\frac{1}{2} \frac{\partial g^{j,k}}{\partial q^i} p_j p_k - \frac{\partial g^{j,N+\beta}}{\partial q^i} p_j \pi_\beta - \frac{1}{2} \frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i} \pi_\beta \pi_\gamma \\ \dot{\pi}_\alpha = -\frac{1}{2} \frac{\partial g^{j,k}}{\partial u^\alpha} p_j p_k - \frac{\partial g^{j,N+\beta}}{\partial u^\alpha} p_j \pi_\beta - \frac{1}{2} \frac{\partial g^{N+\beta,N+\gamma}}{\partial u^\alpha} \pi_\beta \pi_\gamma. \end{cases} \quad (6.17)$$

The conditions  $\gamma_{\mathbf{W}}(0) = (\mathbf{q}_s, \mathbf{u}_s)$ ,  $\gamma_{\mathbf{W}}(1) \in \mathcal{Q} \times \{\mathbf{u}\}$ , and the fact that  $\mathbf{W} \in \Delta_{(\mathbf{q}_s, \mathbf{u}_s)}^\perp$  imply

$$\begin{cases} q^i(0) = 0, \\ u^\alpha(0) = s w_\alpha, \\ u^\alpha(1) = 0. \end{cases} \quad p_i(0) = 0. \quad (6.18)$$

For  $s$  sufficiently small, the existence and uniqueness of the solution to the two-point boundary value problem (6.17)-(6.18) follows from the implicit function theorem. We now seek an expansion of this solution in powers of  $s$ .

Call  $\bar{\pi} = \pi(0)$ , and consider the Cauchy problem for (6.17), with initial data

$$\begin{cases} q(0) = 0, \\ u(0) = s w, \end{cases} \quad \begin{cases} p(0) = 0, \\ \pi(0) = \bar{\pi}. \end{cases} \quad (6.19)$$

Using the Landau order symbols, our computations can be simplified by observing that

$$\begin{cases} q(\sigma) = \mathcal{O}(s^2), \\ p(\sigma) = \mathcal{O}(s^2), \end{cases} \quad \begin{cases} u(\sigma) = \mathcal{O}(s), \\ \pi(\sigma) = \mathcal{O}(s), \end{cases} \quad \text{for all } \sigma \in [0, 1]. \quad (6.20)$$

For all  $\sigma \in [0, 1]$ , the solution of the Cauchy problem (6.17), (6.19) thus satisfies

$$\left\{ \begin{array}{l} q^i(\sigma) = \int_0^\sigma p_i(t) dt + \int_0^\sigma g^{i,N+\beta} \pi_\beta(t) dt + o(s^2), \\ u^\alpha(\sigma) = sw + \int_0^\sigma \pi_\alpha(t) dt + o(s^2), \\ p_i(\sigma) = -\frac{1}{2} \frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0,0) \cdot \int_0^\sigma \pi_\beta(t) \pi_\gamma(t) dt + o(s^2), \\ \pi_\alpha(\sigma) = \bar{\pi} + o(s^2). \end{array} \right. \quad (6.21)$$

From the second and fourth estimates in (6.21) we deduce

$$u^\alpha(\sigma) = sw_\alpha + \sigma \bar{\pi}_\alpha + o(s^2).$$

Since  $u^\alpha(1) = 0$ , this implies

$$\bar{\pi}_\alpha = -sw_\alpha + o(s^2).$$

Using this additional information in the third estimate, we obtain

$$p_i(\sigma) = -\frac{1}{2} \frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0,0) \cdot \sigma s^2 w_\beta w_\gamma + o(s^2),$$

In turn, the first estimate now yields

$$q^i(1) = -\frac{s^2}{4} \frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0,0) w_\beta w_\gamma - \frac{s^2}{2} \frac{\partial g^{i,N+\beta}}{\partial u^\gamma}(0,0) w_\beta w_\gamma + o(s^2).$$

Recalling the identity (6.2), we thus obtain

$$\hat{q}^i(s) \doteq q^i(1) = -\frac{s^2}{2} \frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0,0) w_\beta w_\gamma + o(s^2).$$

In view of property (ii), this establishes (6.16).  $\square$

## 7 A variational characterization of input-output pairs

Let the system  $\Sigma$  be subject to conservative forces generated by a potential  $U : \mathcal{Q} \times \mathcal{U} \mapsto \mathbb{R}$  of class  $\mathcal{U}^1$ . The *Lagrangian* of the system  $(\Sigma, \mathbf{g})$  subject to the potential  $U$  is defined as

$$\mathbb{L}(\mathbf{q}, \mathbf{v}, \mathbf{u}, \mathbf{w}) \doteq \mathcal{T}(\mathbf{q}, \mathbf{u}, \mathbf{v}, \mathbf{w}) + U(\mathbf{q}, \mathbf{u}) \quad ((\mathbf{q}, \mathbf{v}), (\mathbf{u}, \mathbf{w})) \in T\mathcal{Q} \times T\mathcal{U}. \quad (7.1)$$

Moreover, for a given  $(\mathbf{u}, \mathbf{w}) \in T\mathcal{U}$ , we define the  $(\mathbf{u}, \mathbf{w})$ -*Lagrangian*  $\mathbb{L}^{\mathbf{u}, \mathbf{w}}$  by setting

$$\mathbb{L}^{\mathbf{u}, \mathbf{w}}(\mathbf{q}, \mathbf{v}) \doteq \mathbb{L}(\mathbf{q}, \mathbf{v}, \mathbf{u}, \mathbf{w}) \quad (\mathbf{q}, \mathbf{v}) \in T\mathcal{Q}.$$

Let the corresponding  $(\mathbf{u}, \mathbf{w})$ -*Hamiltonian* be defined by

$$\mathbb{H}^{\mathbf{u}, \mathbf{w}}(\mathbf{q}, \mathbf{p}) \doteq (\mathbb{L}^{\mathbf{u}, \mathbf{w}})^*(\mathbf{q}, \mathbf{p}) = \sup_{\mathbf{v} \in T_{\mathbf{q}}\mathcal{Q}} \{ \langle \mathbf{p}, \mathbf{v} \rangle - \mathbb{L}^{\mathbf{u}, \mathbf{w}}(\mathbf{q}, \mathbf{v}) \} \quad (\mathbf{q}, \mathbf{p}) \in T^*\mathcal{Q}.$$

We now show that the admissible input-output pairs, introduced in Definition 3.1, can be characterized by a variational principle.

**Theorem 7.1** Assume that the Lagrangian takes the form (7.1). Let  $u^\sharp : [a, b] \mapsto \mathcal{U}$  be a continuously differentiable control, and let  $t \mapsto (\mathbf{u}^\sharp(t), (\mathbf{q}^\sharp(t), \mathbf{p}^\sharp(t)))$  be an admissible input-output pair, so that (3.12) holds. Then  $q^\sharp(\cdot)$  provides a stationary point to the integral functional

$$\mathcal{I}_{\mathbf{u}^\sharp, \dot{\mathbf{u}}^\sharp}^{[a, b]}(q) \doteq \int_a^b \mathbb{L}^{\mathbf{u}^\sharp(t), \dot{\mathbf{u}}^\sharp(t)}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) dt. \quad (7.2)$$

among all maps  $q(\cdot)$  in the admissible set

$$\mathcal{A} \doteq \left\{ \mathbf{q} : [a, b] \mapsto \mathcal{Q}; \quad q(\cdot) \text{ is absolutely continuous, } \mathbf{q}(a) = \mathbf{q}^\sharp(a), \quad q(b) = \mathbf{q}^\sharp(b) \right\}.$$

In this case, the control equation of motion has a fully Hamiltonian form, namely

$$(\dot{\mathbf{q}}(t), \dot{\mathbf{p}}(t)) = X_{H^{\mathbf{u}(t), \dot{\mathbf{u}}(t)}}(\mathbf{q}(t), \mathbf{p}(t)) \quad (7.3)$$

**Proof.** It is straightforward to check that the control equation of motion (3.12) coincides with the Hamiltonian system (7.3). In turn, this is equivalent to the Euler-Lagrange equations for the integrand in (7.2). Hence, a solution of (3.12) provides a stationary point to the integral functional (7.2).  $\square$

**Remark 7.1** Since the kinetic energy is strictly positive definite, the trajectory  $q^\sharp(\cdot)$  provides a local minimizer. More precisely, there exists  $\delta > 0$  such that the following holds. For every subinterval  $[\tau, \tau'] \subseteq [a, b]$  such that  $\tau' - \tau \leq \delta$ , one has

$$\mathcal{I}_{\mathbf{u}^\sharp, \dot{\mathbf{u}}^\sharp}^{[\tau, \tau']}(q^\sharp) = \min_{q \in \mathcal{A}_{\tau, \tau'}} \mathcal{I}_{\mathbf{u}^\sharp, \dot{\mathbf{u}}^\sharp}^{[\tau, \tau']}(q),$$

where the minimization is taken over the set

$$\mathcal{A}_{\tau, \tau'} \doteq \left\{ \mathbf{q} : [\tau, \tau'] \mapsto \mathcal{Q}; \quad q(\cdot) \text{ is absolutely continuous, } \mathbf{q}(\tau) = \mathbf{q}^\sharp(\tau), \quad q(\tau') = \mathbf{q}^\sharp(\tau') \right\}.$$

## Part II

# Quadratic control systems and stabilizability

## 8 Trajectories of controlled systems with quadratic impulses

We now investigate general control systems of the form:

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_\alpha(x) \dot{u}^\alpha + \sum_{\alpha, \beta=1}^m h_{\alpha, \beta}(x) \dot{u}^\alpha \dot{u}^\beta. \quad (8.1)$$

Here the state variable  $x$  and the control variable  $u$  take values in  $\mathbb{R}^n$  and in  $\mathbb{R}^m$ , respectively. We remark that no a priori bounds are imposed on the derivative  $\dot{u}$ . Our main goal is to understand under which conditions the system can be *stabilized to a given point*  $\bar{x}$ . In

particular, relying on the quadratic dependence on  $\dot{u}$  of the right-hand side of (8.1), in Section 10 we shall investigate *vibrational stabilization*, achieved by means of small periodic oscillations of the control function. In Part III, these results will be applied to the stabilization of the mechanical systems discussed in Part I.

Throughout the following we assume that the functions  $f$ ,  $g_\alpha$ , and  $h_{\alpha,\beta} = h_{\beta,\alpha}$  are at least twice continuously differentiable. We remark that the more general system

$$\dot{x} = \tilde{f}(t, x, u) + \sum_{\alpha=1}^m \tilde{g}_\alpha(t, x, u) \dot{u}^\alpha + \sum_{\alpha,\beta=1}^m \tilde{h}_{\alpha,\beta}(t, x, u) \dot{u}^\alpha \dot{u}^\beta,$$

where the vector fields depend also on time and on the control  $u$ , can be easily rewritten in the form (8.1). Indeed, it suffices to work in the extended state space  $x \in \mathbb{R}^{1+n+m}$ , introducing the additional state variables  $x^0 = t$  and  $x_{n+\alpha} = u^\alpha$ , with equations

$$\dot{x}^0 = 1, \quad \dot{x}_{n+\alpha} = \dot{u}^\alpha \quad \alpha = 1, \dots, m.$$

Given the initial condition

$$x(0) = \tilde{x}, \quad (8.2)$$

for every smooth control function  $u : [0, T] \mapsto \mathbb{R}^m$  one obtains a unique solution  $t \mapsto x(t; u)$  of the Cauchy problem (8.1)-(8.2). More generally, since the equation (8.1) is quadratic w.r.t. the derivative  $\dot{u}$ , it is natural to consider admissible controls in a set of absolutely continuous functions  $u(\cdot)$  with derivatives in  $\mathbf{L}^2$ . For example, for a given  $K > 0$ , one could allow the controls to belong to

$$\left\{ u : [0, T] \mapsto \mathbb{R}^m; \quad \int_0^T |\dot{u}(t)|^2 dt \leq K \right\}. \quad (8.3)$$

The main goal of the following analysis is to provide a characterization of the closure of this set of trajectories, in terms of an auxiliary differential inclusion. Let us notice that the system (8.1) is naturally connected with the differential inclusion

$$\dot{x} \in \mathcal{F}(x), \quad (8.4)$$

where, for every  $x \in \mathbb{R}^n$ ,

$$\mathcal{F}(x) \doteq \overline{\text{co}} \left\{ f(x) + \sum_{\alpha=1}^m g_\alpha(x) w^\alpha + \sum_{\alpha,\beta=1}^m h_{\alpha,\beta}(x) w^\alpha w^\beta; \quad (w^1, \dots, w^m) \in \mathbb{R}^m \right\}. \quad (8.5)$$

Here and in the sequel, for any given subset  $A$  of a topological vector space,  $\overline{\text{co}}A$  denotes the closed convex hull of  $A$ .

In addition, it will be convenient to work also in an extended state space, using the variable  $\hat{x} = \begin{pmatrix} x^0 \\ x \end{pmatrix} \in \mathbb{R}^{1+n}$ . For a given  $\hat{x}$ , consider the set

$$F(\hat{x}) \doteq \overline{\text{co}} \left\{ \begin{pmatrix} 1 \\ f(x) \end{pmatrix} (a^0)^2 + \sum_{\alpha=1}^m \begin{pmatrix} 0 \\ g_\alpha(x) \end{pmatrix} a^0 a^\alpha + \sum_{\alpha,\beta=1}^m \begin{pmatrix} 0 \\ h_{\alpha,\beta}(x) \end{pmatrix} a^\alpha a^\beta; \quad a^0 \in [0, 1], \quad \sum_{\alpha=0}^m (a^\alpha)^2 = 1 \right\}. \quad (8.6)$$

Notice that  $F$  is a convex, compact valued multifunction on  $\mathbb{R}^{1+n}$ , Lipschitz continuous w.r.t. the Hausdorff metric [2].

For a given interval  $[0, S]$ , the set of trajectories of the *graph differential inclusion*

$$\frac{d}{ds}\hat{x}(s) \in F(\hat{x}(s)), \quad \hat{x}(0) = \begin{pmatrix} 0 \\ x^\# \end{pmatrix} \quad (8.7)$$

is a non-empty, closed, bounded subset of  $\mathcal{C}([0, S]; \mathbb{R}^{1+n})$ . Consider one particular solution, say  $s \mapsto \hat{x}(s) = \begin{pmatrix} x^0(s) \\ x(s) \end{pmatrix}$ , defined for  $s \in [0, S]$ . Assume that  $T \doteq x^0(S) > 0$ . Since the map  $s \mapsto x^0(s)$  is non-decreasing, it admits a generalized inverse

$$s = s(t) \quad \text{iff} \quad x^0(s) = t. \quad (8.8)$$

Indeed, for all but countably many times  $t \in [0, T]$  there exists a unique value of the parameter  $s$  such that the identity on the right of (8.8) holds. We can thus define a corresponding trajectory

$$t \mapsto x(t) = x(s(t)) \in \mathbb{R}^n. \quad (8.9)$$

This map is well defined for almost all times  $t \in [0, T]$ .

To establish a connection between the original control system (8.1) and the differential inclusion (8.7), consider first a smooth control function  $u(\cdot)$ . As in [26], we define a reparametrized time variable by setting

$$s(t) \doteq \int_0^t \left(1 + \sum_{\alpha=1}^m (\dot{u}^\alpha)^2(\tau)\right) d\tau. \quad (8.10)$$

Notice that the map  $t \mapsto s(t)$  is strictly increasing. The inverse map  $s \mapsto t(s)$  is uniformly Lipschitz continuous and satisfies

$$\frac{dt}{ds} = \left(1 + \sum_{\alpha=1}^m (\dot{u}^\alpha)^2(t)\right)^{-1}.$$

Let now  $x : [0, T] \mapsto \mathbb{R}^n$  be a solution of (8.1) corresponding to the smooth control  $u : [0, T] \mapsto \mathbb{R}^m$ . We claim that the map  $s \mapsto \hat{x}(s) \doteq \begin{pmatrix} t(s) \\ x(t(s)) \end{pmatrix}$  is a solution to the differential inclusion (8.7). Indeed, setting

$$a^0(s) \doteq \frac{1}{\sqrt{1 + \sum_{\beta=1}^m (\dot{u}^\beta)^2(t(s))}}, \quad a^\alpha(s) \doteq \frac{\dot{u}^\alpha(t(s))}{\sqrt{1 + \sum_{\beta=1}^m (\dot{u}^\beta)^2(t(s))}} \quad \alpha = 1, \dots, m, \quad (8.11)$$

one has

$$\begin{cases} \frac{dt}{ds} = (a^0)^2(s) \\ \frac{dx}{ds} = f(x(s)) (a^0)^2(s) + \sum_{\alpha=1}^m g_\alpha(x(s)) a^0(s) a^\alpha(s) + \sum_{\alpha, \beta=1}^m h_{\alpha, \beta}(x(s)) a^\alpha(s) a^\beta(s). \end{cases} \quad (8.12)$$

Hence  $\hat{x}(\cdot) = (t(\cdot), x(\cdot))$  verifies (8.7), because, by (8.11),

$$a^0(s) \in [0, 1], \quad \sum_{\alpha=0}^m (a^\alpha)^2(s) \equiv 1.$$

Notice that the derivatives  $\dot{u}^\alpha$  can now be recovered as

$$\dot{u}^\alpha(t) = \frac{a^\alpha(s(t))}{a^0(s(t))} \quad \alpha = 1, \dots, m. \quad (8.13)$$

The following theorem shows that every solution of the differential inclusion (8.7) can be approximated by smooth solutions of the original control system (8.1).

**Theorem 8.1** *Let  $\hat{x} = (x^0, x) : [0, S] \mapsto \mathbb{R}^{1+n}$  be a solution to the multivalued Cauchy problem (8.7) such that  $x^0(S) = T > 0$ . Then there exists a sequence of smooth control functions  $u_\nu : [0, T] \mapsto \mathbb{R}^M$  such that the corresponding solutions*

$$s \mapsto \hat{x}_\nu(s) = \begin{pmatrix} t_\nu(s) \\ x_\nu(s) \end{pmatrix}$$

*of the equations (8.11)-(8.12) converge to the map  $s \mapsto \hat{x}(s)$  uniformly on  $[0, S]$ . Moreover, defining the function  $x(t) = x(s(t))$  as in (8.9), we have*

$$\lim_{\nu \rightarrow \infty} \int_0^T |x(t) - x_\nu(t)| dt = 0. \quad (8.14)$$

**Proof.** By the assumption, the extended vector fields

$$\hat{f} = \begin{pmatrix} 1 \\ f \end{pmatrix}, \quad \hat{g}_\alpha = \begin{pmatrix} 0 \\ g_\alpha \end{pmatrix}, \quad \hat{h}_{\alpha,\beta} = \begin{pmatrix} 0 \\ h_{\alpha,\beta} \end{pmatrix}$$

are Lipschitz continuous. Consider the set of trajectories of the control system

$$\frac{d}{ds} \hat{x} = \hat{f} \cdot (a^0)^2 + \sum_{\alpha=1}^m \hat{g}_\alpha a^0 a^\alpha + \sum_{\alpha,\beta=1}^m \hat{h}_{\alpha,\beta} a^\alpha a^\beta, \quad \hat{x}(0) = \begin{pmatrix} 0 \\ x^\# \end{pmatrix}, \quad (8.15)$$

where the controls  $a = (a^0, a^1, \dots, a^m)$  satisfy the pointwise constraints

$$a^0(s) \in [0, 1], \quad \sum_{\alpha=0}^m (a^\alpha)^2(s) = 1 \quad s \in [0, S]. \quad (8.16)$$

In the above setting, it is well known [2] that the set of trajectories

$$s \mapsto \hat{x}(s) = (x^0, x^1, \dots, x^n)(s)$$

of (8.15)-(8.16) is dense on the set of solutions to the differential inclusion (8.7). Hence there exists a sequence of control functions  $s \mapsto a_\nu(s) = (a_\nu^0, \dots, a_\nu^m)(s)$ ,  $\nu \geq 1$ , such that the corresponding solutions  $s \mapsto \hat{x}_\nu(s)$  of (8.15) converge to  $\hat{x}(\cdot)$  uniformly for  $s \in [0, S]$ . In particular, this implies the convergence of the first components:

$$x_\nu^0(S) = \int_0^S [a_\nu^0(s)]^2 ds \rightarrow x^0(S) = T. \quad (8.17)$$

We now observe that the “input-output map”  $a(\cdot) \mapsto \hat{x}(\cdot, a)$  from controls to trajectories is uniformly continuous as a map from  $\mathbf{L}^1([0, S]; \mathbb{R}^{1+m})$  into  $\mathcal{C}([0, S]; \mathbb{R}^{1+n})$ . By slightly

modifying the controls  $a_\nu$  in  $\mathbf{L}^1$ , we can replace the sequence  $a_\nu$  by a new sequence of smooth control functions  $\tilde{a}_\nu : [0, S] \mapsto \mathbb{R}^{1+m}$  with the following properties:

$$\tilde{a}_\nu^0(s) > 0 \quad \text{for all } s \in [0, S], \quad \nu \geq 1. \quad (8.18)$$

$$\int_0^S [\tilde{a}_\nu^0(s)]^2 ds = T \quad \text{for all } \nu \geq 1, \quad (8.19)$$

$$\lim_{\nu \rightarrow \infty} \int_0^S |\tilde{a}_\nu(s) - a_\nu(s)| ds = 0. \quad (8.20)$$

This implies the uniform convergence

$$\lim_{\nu \rightarrow \infty} \|\hat{x}(\cdot, \tilde{a}_\nu) - \hat{x}(\cdot)\|_{\mathcal{C}([0, S]; \mathbb{R}^{1+n})} = 0. \quad (8.21)$$

By (8.18), for each  $\nu \geq 1$  the map

$$s \mapsto x_\nu^0(s) \doteq \int_0^s [\tilde{a}_\nu^0(s)]^2 ds$$

is strictly increasing. Therefore it has a smooth inverse  $s = s_\nu(t)$ . Recalling (8.13), we now define the sequence of smooth control functions  $u_\nu : [0, T] \mapsto \mathbb{R}^m$  by setting  $u_\nu(t) = (u_\nu^1, \dots, u_\nu^m)(t)$ , with

$$u_\nu^\alpha(t) = \int_0^t \frac{\tilde{a}_\nu^\alpha(s_\nu(\tau))}{\tilde{a}_\nu^0(s_\nu(\tau))} d\tau. \quad (8.22)$$

By construction, the solutions  $t \mapsto x_\nu(t; u_\nu)$  of the original system (8.1) corresponding to the controls  $u_\nu$  coincide with the trajectories  $t \mapsto (\hat{x}_\nu^1, \dots, \hat{x}_\nu^n)(s_\nu(t))$ , where  $\hat{x}_\nu = (\hat{x}_\nu^0, \hat{x}_\nu^1, \dots, \hat{x}_\nu^n)$  is the solution of (8.15) with control  $\tilde{a}_\nu = (\tilde{a}_\nu^0, \dots, \tilde{a}_\nu^m)$ .

To prove the last statement in the theorem, define the increasing functions

$$t(s) = \int_0^s [\tilde{a}_\nu^0(r)]^2 dr, \quad t_\nu(s) = \int_0^s [\tilde{a}_\nu^0(r)]^2 dr,$$

and let  $t \mapsto s(t)$ ,  $t \mapsto s_\nu(t)$  be their inverses, respectively. Notice that each  $s_\nu(\cdot)$  is smooth. Moreover,

$$\left| \frac{d}{ds} t(s) \right| \leq 1, \quad \left| \frac{d}{ds} t_\nu(s) \right| \leq 1, \quad (8.23)$$

$$\lim_{\nu \rightarrow \infty} \int_0^T |s(t) - s_\nu(t)| dt = \lim_{\nu \rightarrow \infty} \int_0^S |t(s) - t_\nu(s)| ds = 0. \quad (8.24)$$

Using (8.23), we obtain the estimate

$$\begin{aligned} \int_0^T |x(t) - x_\nu(t)| dt &= \int_0^T |x(s(t)) - x_\nu(s(t))| dt + \int_0^T |x_\nu(s(t)) - x_\nu(s_\nu(t))| dt \\ &\leq \int_0^S |x(s) - x_\nu(s)| ds + C \cdot \int_0^T |s(t) - s_\nu(t)| dt. \end{aligned} \quad (8.25)$$



Here the constant  $C$  denotes an upper bound for the derivative w.r.t.  $s$ , for example

$$C \doteq \sup_x \left\{ |f(x)| + \sum_i |g_\alpha(x)| + \sum_{\alpha,\beta} |h_{\alpha,\beta}(x)| \right\}, \quad (8.26)$$

where the supremum is taken over a compact set containing the graphs of all functions  $x_\nu(\cdot)$ . By (8.21) and (8.24), the right hand side of (8.25) vanishes in the limit  $\nu \rightarrow \infty$ . This completes the proof of the theorem.  $\square$

**Remark 8.1** For a given time interval  $[0, T]$ , we are considering controls  $u(\cdot)$  in the Sobolev space  $W^{1,2}$ . The corresponding solutions are absolutely continuous maps, namely they belong to  $W^{1,1}$ . Now consider a sequence of control functions  $u_\nu$ , whose derivatives are uniformly bounded in  $L^2$ . Assume that the corresponding reparametrized trajectories  $s \mapsto (t_\nu(s), x_\nu(s))$ , constructed as in (8.11)-(8.12), converge to a path  $s \mapsto (t(s), x(s))$ , providing a solution to (8.7). We wish to point out that, in general, the projection on the state space  $t \mapsto x(s(t))$  *may well be discontinuous*. Notice that, on the contrary, the uniform limit of the controls  $t \mapsto u_\nu(t)$  must be Hölder continuous, because of the uniform  $\mathbf{L}^2$  bound on the derivatives.

A completely different situation arises when all the vector fields  $h_{\alpha,\beta}$  vanish identically, so that (8.1) reduces to

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_\alpha(x) \dot{u}^\alpha \quad (8.27)$$

Systems of this form have been extensively studied, see [33], [20], [5], [6], or the surveys [24], [3] and the references therein. In this case, solutions can be well defined also for general control functions  $u(\cdot)$  with bounded variation but possibly discontinuous. We recall that, unless the Lie brackets  $[g_\alpha, g_\beta]$  vanish identically, one needs to assign a “graph completion” of the control  $u(\cdot)$  in order to determine uniquely the trajectory. Indeed, at each time  $\tau$  where  $u$  has a jump, one should also specify a continuous path joining the left state  $u(\tau-)$  with the right state  $u(\tau+)$ . See [5] for details.

**Remark 8.2** Assume again that system (8.1) reduces to (8.27), and consider a sequence of Lipschitz controls  $u_\nu$  having equi-bounded derivatives on compact sets and converging to a continuous control  $\tilde{u}$  uniformly on bounded sets. Moreover, consider initial states  $x_\nu^\sharp$  converging to a point  $x^\sharp$ . Then, the trajectories  $x_\nu$  corresponding to the controls  $u_\nu$  and the initial conditions  $x_\nu(0) = x_\nu^\sharp$  converge to the solution  $\tilde{x}$  of (8.27) corresponding to the control  $\tilde{u}$  and initial condition  $\tilde{x}(0) = x^\sharp$ . In particular, if  $\tilde{u}$  is constant, the  $x_\nu$  converge to the solution of

$$\dot{x} = f(x) \quad x(0) = x^\sharp \quad (8.28)$$

## 9 Stabilization

In this section we examine various concepts of stability for the impulsive system (8.1) and relate them to the weak stability of the differential inclusion (8.6)-(8.7).

**Definition 9.1** *We say that the control system (8.1) is stabilizable at the point  $\bar{x} \in \mathbb{R}^n$  if, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that the following holds. For every initial state  $x^\sharp$  with*

$|x^\sharp - \bar{x}| \leq \delta$  there exists a smooth control function  $t \mapsto u(t) = (u^1, \dots, u^m)(t)$  such that the corresponding trajectory of (8.1)-(8.2) satisfies

$$|x(t, u) - \bar{x}| \leq \varepsilon \quad \text{for all } t \geq 0. \quad (9.1)$$

Any such control will be called a stabilizing control

We say that the system (8.1) is asymptotically stabilizable at the point  $\bar{x}$  if a control  $u(\cdot)$  can be found such that, in addition to (9.1), there holds

$$\lim_{t \rightarrow \infty} x(t, u) = \bar{x}. \quad (9.2)$$

Any such control will be called an asymptotically stabilizing control.

**Remark 9.1** Notice that the point  $\bar{x}$  needs not to be an equilibrium point for the vector field  $f$ .

**Remark 9.2** We require here that the stabilizing controls be smooth. As it will become apparent in the sequel, this is hardly a restriction. Indeed, in all cases under consideration, if a stabilizing control  $u \in W^{1,2}$  is found, by approximation one can construct a smooth control  $\tilde{u}$  which is still stabilizing.

**Remark 9.3** In the above definitions we are not putting any constraint on the control function  $u : [0, \infty[ \mapsto \mathbb{R}^m$ . In principle, one may well have  $|u(t)| \rightarrow \infty$  as  $t \rightarrow \infty$ . If one wishes to stabilize the system (8.1) and at the same time keep the control values within a small neighborhood of a given value  $\bar{u}$ , it suffices to consider the stabilization problem for an augmented system, adding the variables  $x^{n+1}, \dots, x^{n+m}$  together with the equations

$$\dot{x}^{n+\alpha} = \dot{u}^\alpha \quad \alpha = 1, \dots, m.$$

Similar stability concepts can be also defined for the differential inclusion

$$\dot{x} \in F(x), \quad (9.3)$$

see for example [29]. We recall that a trajectory of (9.3) is an absolutely continuous function  $t \mapsto x(t)$  which satisfies the differential inclusion at a.e. time  $t$ .

**Definition 5.2.** The point  $\bar{x}$  is *weakly stable* for the differential inclusion (9.3) if, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that the following holds. For every initial state  $x^\sharp$  with  $|x^\sharp - \bar{x}| \leq \delta$  there exists a trajectory  $x(\cdot)$  of (9.3) such that

$$x(0) = x^\sharp, \quad |x(t) - \bar{x}| \leq \varepsilon \quad \text{for all } t \geq 0. \quad (9.4)$$

Moreover,  $\bar{x}$  is *weakly asymptotically stable* if, there exists a trajectory which, in addition to (9.4), satisfies

$$\lim_{t \rightarrow \infty} x(t) = \bar{x}. \quad (9.5)$$

In connection with the multifunction  $F$  defined at (8.6), we consider a second multifunction  $F^\diamond$  obtained by projecting the sets  $F(\hat{x}) \subset \mathbb{R}^{1+n}$  into the subspace  $\mathbb{R}^n$ . More precisely, we set

$$F^\diamond(x) \doteq \overline{\text{co}} \left\{ f(x) (a^0)^2 + \sum_{\alpha=1}^m g_\alpha(x) a^0 a^\alpha + \sum_{\alpha,\beta=1}^m h_{\alpha,\beta}(x) a^\alpha a^\beta ; \quad w^0 \in [0, 1], \quad \sum_{\alpha=0}^m (w^\alpha)^2 = 1 \right\}. \quad (9.6)$$

Observe that, if the vector fields  $f, g_\alpha$ , and  $h_{\alpha,\beta}$  are Lipschitz continuous, then the multifunction  $F^\diamond$  is Lipschitz continuous with compact, convex values. Our first result in this section is:

**Theorem 9.1** *The impulsive system (8.1) is asymptotically stabilizable at the point  $\bar{x}$  if and only if  $\bar{x}$  is weakly asymptotically stable for the projected graph differential inclusion*

$$\frac{d}{ds} x(s) \in F^\diamond(x(s)). \quad (9.7)$$

**Proof.** Let  $\bar{x}$  be weakly asymptotically stable for (9.7). Without loss of generality, we can assume  $\bar{x} = 0$ .

Given  $\varepsilon > 0$ , choose  $\delta > 0$  such that, if  $|x^\sharp| \leq \delta$ , then there exists a trajectory  $t \mapsto x(s)$  of the differential inclusion (9.7) such that  $x(0) = x^\sharp$ ,  $|x(s)| \leq \varepsilon/2$  for all  $t \geq 0$  and  $x(s) \rightarrow 0$  as  $t \rightarrow \infty$ . Using the basic approximation property stated in Theorem 8.1, we will construct a smooth control  $t \mapsto u(t) = (u^1, \dots, u^m)(t)$  such that the corresponding trajectory  $x(\cdot; u)$  of (8.1)-(8.2) satisfies

$$|x(t)| \leq \varepsilon \quad \text{for all } t \geq 0, \quad \lim_{t \rightarrow \infty} x(t) = 0. \quad (9.8)$$

Define the decreasing sequence of positive numbers  $\varepsilon_k \doteq \varepsilon 2^{-k}$ . For each  $k \geq 0$ , choose  $\delta_k > 0$  so that, whenever  $|x^\sharp| \leq \delta_k$ , there exists a solution to (9.7) with

$$x(0) = x^\sharp, \quad \lim_{s \rightarrow \infty} x(s) = 0, \quad |x(s)| < \frac{\varepsilon_k}{2} \quad \text{for all } s \geq 0. \quad (9.9)$$

Choose a sequence of strictly positive integers  $k(1) \leq k(2) \leq \dots$ , such that

$$\lim_{j \rightarrow \infty} k(j) = \infty, \quad \sum_{j=1}^{\infty} \delta_{k(j)} = \infty. \quad (9.10)$$

Note that the second condition in (9.10) is certainly satisfied if the numbers  $k(j)$  grow at a sufficiently slow rate.

Assume  $|x^\sharp| \leq \delta_0$ . A smooth control  $u$  steering the system (8.1) from  $x^\sharp$  asymptotically toward the origin will be constructed by induction on  $j$ . For  $j = 1$ , let  $x : [0, s_1] \mapsto \mathbb{R}^n$  be a trajectory of the differential inclusion (9.7) such that

$$x(0) = x^\sharp, \quad |x(s_1)| < \frac{\delta_{k(1)}}{3}, \quad |x(s)| < \frac{\varepsilon_0}{2} \quad \text{for all } s \in [0, s_1].$$

By the definition of  $F^\diamond$ , there exists a trajectory of the differential inclusion (8.7) having the form  $s \mapsto \hat{x}(s) = (x^0(s), x(s))$ . Notice that, in order to apply Theorem 8.1 and approximate

$x(\cdot)$  with a smooth solution of the control system (8.1) we would need  $x^0(s_1) > 0$ . This is not yet guaranteed by the above construction. To take care of this problem, we define  $s'_1 \doteq s_1 + \delta_{k(1)}/3C$ , where  $C$  provides a local upper bound for the magnitude of the vector field  $f$ , as in (8.26). We then prolong the trajectory  $\hat{x}(\cdot)$  to the larger interval  $[0, s'_1]$ , by setting

$$\frac{d}{ds} \begin{pmatrix} x^0(s) \\ x(s) \end{pmatrix} = \begin{pmatrix} 1 \\ f(x) \end{pmatrix} \quad s \in ]s_1, s'_1].$$

This construction achieves the inequalities

$$x^0(s'_1) \geq s'_1 - s_1 \geq \frac{\delta_{k(1)}}{3C}, \quad |x(s'_1)| < \frac{2}{3}\delta_{k(1)}.$$

Set  $\tau^1 \doteq x^0(s'_1)$ . By Theorem 8.1, there exists a smooth control  $u : [0, \tau^1] \mapsto \mathbb{R}^m$  such that the corresponding solution  $s \mapsto (x^0(s, u), x(s, u))$  of (8.11)-(8.12) differs from the above trajectory by less than  $\delta_{k(1)}/3$ , namely

$$|x^0(s, u) - x^0(s)| < \frac{\delta_{k(1)}}{3}, \quad |x(s; u) - x(s)| < \frac{\delta_{k(1)}}{3} \quad \text{for all } s \in [0, s'_1].$$

In particular, setting  $x(t, u) \doteq x(s(t), u)$  as in (8.9), this implies

$$|x(\tau_1, u)| < \delta_{k(1)}, \quad |x(t, u)| < \frac{\varepsilon_0}{2} + \frac{\delta_{k(1)}}{3} \leq \varepsilon_0 \quad \text{for all } t \in [0, \tau_1].$$

The construction now proceeds by induction on  $j$ . Assume that a smooth control  $u(\cdot)$  has been constructed on the time interval  $[0, \tau_j]$ , in such a way that

$$|x(\tau_j, u)| < \delta_{k(j)}, \quad |x(t, u)| < \varepsilon_{k(j-1)} \quad \text{for all } t \in [\tau_{j-1}, \tau_j]. \quad (9.11)$$

By assumptions, there exists a trajectory  $s \mapsto x(s)$  of the differential inclusion (9.7) such that

$$x(0) = x(\tau_j, u), \quad |x(s_j)| < \frac{\delta_{k(j+1)}}{3}, \quad |x(s)| < \frac{\varepsilon_{k(j)}}{2} \quad \text{for all } s \in [0, s_j]. \quad (9.12)$$

This trajectory is extended to the slightly larger interval  $[0, s'_j]$ , with  $s'_j = s_j + \delta_{k(j)}/3C$ , by setting

$$\frac{d}{ds} \begin{pmatrix} x^0(s) \\ x(s) \end{pmatrix} = \begin{pmatrix} 1 \\ f(x) \end{pmatrix} \quad s \in ]s_j, s'_j]. \quad (9.13)$$

Notice that, by (9.12), (9.13), and (8.26), we have

$$x^0(s'_j) \geq s'_j - s_j \geq \frac{\delta_{k(j)}}{3C}, \quad |x(s'_j)| < \frac{2}{3}\delta_{k(j+1)}. \quad (9.14)$$

Set  $\tau_{j+1} \doteq \tau_j + x^0(s'_j)$ . Using again Theorem 8.1, we can extend the control  $u : [0, \tau_j] \mapsto \mathbb{R}^M$  to a continuous, piecewise smooth control defined on the larger interval  $[0, \tau_{j+1}]$ , such that the corresponding solution  $s \mapsto x(s, u)$  of (8.1)-(8.2) satisfies

$$|x(\tau_{j+1}, u)| < \delta_{k(j+1)}, \quad |x(t, u)| < \varepsilon_{k(j)} \quad \text{for all } t \in [\tau_j, \tau_{j+1}]. \quad (9.15)$$

Notice that, at this stage, the control  $u$  is obtained by piecing together two smooth control functions, defined on the intervals  $[0, \tau_j]$  and  $[\tau_j, \tau_{j+1}]$  respectively. This makes  $u$  continuous

but possibly not  $\mathcal{C}^1$  in a neighborhood of the point  $\tau_j$ . To fix this problem, we slightly modify the values of  $u$  in a small neighborhood of  $\tau_j$ , so that  $u$  becomes smooth also at this point, while the strict inequalities (9.15) still hold.

Having completed the inductive steps for all  $j \geq 1$  we observe that

$$\lim_{j \rightarrow \infty} \tau_j = \sum_j \frac{\delta_{k(j)}}{3C} = \infty$$

because of (9.10). As  $t \rightarrow \infty$ , by (9.15) we have  $x(t, u) \rightarrow 0$ . This shows that the impulsive system (8.1) is asymptotically stabilizable at the origin, proving one of the implications stated in the theorem.

The converse implication is obvious, because every solution of the system (8.1) corresponding to a smooth control yields a solution to the differential inclusion (9.7), after a suitable time rescaling.

**Corollary 9.1** *Let a point  $\bar{x}$  be weakly asymptotically stable for the differential inclusion (8.4). Then the system (8.1) is asymptotically stabilizable at  $\bar{x}$ .*

**Proof.** Since the point  $\bar{x}$  is weakly asymptotically stable for (8.4), then it is asymptotically stable for the differential inclusion (9.7), which, in turn, implies that the impulsive system (8.1) can be stabilized at  $\bar{x}$ .

## 9.1 Lyapunov functions

There is an extensive literature, in the context of O.D.E's and of control systems or differential inclusions, relating the stability of an equilibrium state to the existence of a Lyapunov function. We recall below the basic definition, in a form suitable for our applications. For simplicity, we henceforth consider the case  $\bar{x} = 0 \in \mathbb{R}^n$ , which of course is not restrictive.

**Definition 9.2** *Let  $x \mapsto G(x) \subset \mathbb{R}^n$  be a set valued function defined for  $x \in \mathbb{R}^n$ . A scalar function  $V$  defined on a neighborhood  $\mathcal{N}$  of the origin is a weak Lyapunov function for the differential inclusion*

$$\dot{x} \in G(x)$$

*if the following holds.*

- (i)  $V$  is continuous on  $\mathcal{N}$ , and continuously differentiable on  $\mathcal{N} \setminus \{0\}$ .
- (ii)  $V(0) = 0$  while  $V(x) > 0$  for all  $x \neq 0$ ,
- (iii) For each  $\delta > 0$  sufficiently small, the sublevel set  $\{x; V(x) \leq \delta\}$  is compact.
- (iv) At each  $x \neq 0$  one has

$$\inf_{y \in G(x)} \nabla V(x) \cdot y \leq 0. \tag{9.16}$$

The following theorem relates the stability of the impulsive control system (8.1) to the existence of a Lyapunov function for the differential inclusion (8.4).

**Theorem 9.2** *Consider the multifunction  $\mathcal{F}$  defined at (8.5). Assume that the differential inclusion (8.4) admits a Lyapunov function  $V = V(x)$  defined on a neighborhood  $\mathcal{N}$  of the origin. Then the control system (8.1) can be stabilized at the origin.*

**Remark 9.4** We are here requiring that the function  $V$  satisfies the conditions (i)–(iii) in Definition 9.2, and that for each  $x \neq 0$  there exists  $z \in \mathcal{F}(x)$  such that

$$\nabla V(x) \cdot z \leq 0. \quad (9.17)$$

Notice that the multifunction  $\mathcal{F}$  in (8.5) has unbounded values. An equivalent condition, formulated in terms of the bounded multifunction  $F$  in (8.6) is the following.

(iv') *For every  $x \in \mathcal{N} \setminus \{0\}$ , there exists  $\hat{y} = (y_0, y) \in F(x)$  such that*

$$\nabla V(x) \cdot y \leq 0 \quad y_0 > 0. \quad (9.18)$$

Notice that the set of conditions (i)–(iii) and (iv') represents a slight strengthening of the notion of weak Lyapunov function when this is applied to the projected graph differential equation (9.7). Yet, let us point out that the weak stability of (9.7) is not enough to guarantee the bility of the control system (8.1), so the condition  $y_0 > 0$  in (9.18) plays a crucial role. Indeed, on  $\mathbb{R}^2$ , consider the constant vector fields  $f = (1, 0)$ ,  $h_{11} = (0, 1)$ ,  $h_{22} = (0, -1)$ ,  $g_1 = g_2 = h_{12} = h_{21} = (0, 0)$ . Then, choosing  $a^0 = 0$ ,  $a^1 = a^2 = 1/\sqrt{2}$  we see that  $(0, 0, 0) \in F(x)$  for every  $x \in \mathbb{R}^2$ . Hence condition

$$\inf_{y \in F(x)} \nabla V \cdot y \leq 0$$

is trivially satisfied by any function  $V$ . However, it is clear that in this case the system (8.1) is not stabilizable at the origin.

**Remark 9.5** Theorem 9.2 is somewhat weaker than its counterpart, Theorem 9.1, dealing with asymptotic stability. Indeed, to prove that the impulsive control system (8.1) is stabilizable, we need to assume not only that the differential inclusion (9.7) is weakly stable, but also that there exists a Lyapunov function.

**Proof of Theorem 9.2.** Given  $\varepsilon > 0$ , choose  $\delta > 0$  such that

$$V(x) \leq 2\delta \quad \text{implies} \quad |x| \leq \varepsilon.$$

Let an initial state  $x^\sharp$  be given, with  $V(x^\sharp) \leq \delta$ .

According to Remark 9.4, for every  $x \neq 0$  there exists  $(y_0, y) \in F(x)$  such that (9.18) holds. We recall that the multifunction  $F$  in (8.6) is Lipschitz continuous, with compact, convex values. Since the set  $\Omega \doteq \{x; \delta \leq V(x) \leq 3\delta\}$  is compact, by the continuity of  $\nabla V$  we can find  $\kappa > 0$  such that, for every  $x \in \Omega$ , there exists  $\hat{y} = (y_0, y) \in F(x)$  with

$$\nabla V(x) \cdot y \leq 0, \quad y_0 \geq \kappa.$$

The control  $u$  will be defined inductively on a sequence of the time intervals  $[\tau_{j-1}, \tau_j]$ , with  $\tau_j \geq j\kappa$ . Set  $\tau_0 = 0$ . Consider the differential inclusion

$$\frac{d}{ds}\hat{x}(s) \in \begin{cases} F(x(s)) \cap \{(y_0, y); \nabla V(x) \cdot y \leq 0, y_0 \geq \kappa\} & \text{if } \delta < V(x) < 2\delta, \\ F(x(s)) & \text{if } V(x) \leq \delta \text{ or } V(x) \geq 2\delta, \end{cases} \quad (9.19)$$

with initial data  $\hat{x}(0) = (0, x^\#)$ . The right-hand side of (9.19) is an upper semicontinuous multifunction, with nonempty compact convex values. Therefore (see for example [2]), the Cauchy problem admits at least one solution  $s \mapsto \hat{x}(s) = (x^0(s), x(s))$ , defined for  $s \in [0, 1]$ . We observe that this solution satisfies

$$x^0(1) \geq \kappa, \quad V(x(s)) \leq \delta \quad \text{for all } s \in [0, 1].$$

Hence, by Theorem 8.1 there exists a smooth control  $u : [0, \tau_1] \mapsto \mathbb{R}^m$ , with  $\tau_1 = x^0(1) \geq \kappa$ , such that the corresponding trajectory of (8.1)-(8.2) satisfies

$$V(x(t, u)) < \frac{3}{2}\delta = 2\delta - 2^{-1}\delta \quad \text{for all } t \in [0, \tau_1].$$

By induction, assume now that a smooth control  $u(\cdot)$  has been constructed on the interval  $[0, \tau_j]$  with  $\tau_j \geq \kappa j$ , and that the corresponding trajectory  $t \mapsto x(t, u)$  of the impulsive system (8.1)-(8.2) satisfies

$$V(x(t, u)) \leq 2\delta - 2^{-j}\delta \quad t \in [0, \tau_j]. \quad (9.20)$$

We then construct a solution  $s \mapsto \hat{x}(s) = (x^0(s), x(s))$  of the differential inclusion (9.19) for  $s \in [0, 1]$ , with initial data  $\hat{x}(0) = (0, x(\tau_j, u))$ . This function will satisfy

$$x^0(1) \geq \kappa, \quad V(x(s)) < 2\delta - 2^{-j}\delta \quad \text{for all } s \in [0, 1].$$

Using again Theorem 8.1, we can prolong the control  $u$  to a larger time interval  $[0, \tau_{j+1}]$ , with  $\tau_{j+1} - \tau_j = x^0(1) \geq \kappa$ , in such a way that

$$V(x(t, u)) < 2\delta - 2^{-j-1}\delta \quad t \in [0, \tau_{j+1}]. \quad (9.21)$$

At a first stage, this control  $u$  will be piecewise smooth, continuous but not  $\mathcal{C}^1$  in a neighborhood of the point  $\tau_j$ . By a local approximation, we can slightly change its values in a small neighborhood of the point  $\tau_j$ , making it smooth also at the point  $\tau_j$ , and preserving the strict inequalities (9.21).

Since  $\tau_j \geq \kappa j$  for all  $j \geq 1$ , as  $j \rightarrow \infty$  the induction procedure generates a smooth control function  $u(\cdot)$ , defined for all  $t \geq 0$ , whose corresponding trajectory satisfies  $V(x(t, u)) < 2\delta$  for all  $t \geq 0$ . This completes the proof of the theorem.  $\square$

## 10 A selection technique

In the previous section we proved two general results, relating the stability of the control system (8.1) to the weak stability of the differential inclusion (8.4). A complete description of the sets  $\mathcal{F}(x)$  in (8.5) may often be very difficult. However, as shown in [29], to establish a stability property it suffices to construct a suitable family of smooth selections. We shall briefly describe this approach.

Let a point  $\bar{x} \in \mathbb{R}^n$  be given, and assume that there exists a  $\mathcal{C}^1$  selection

$$\gamma(x, \xi) \in \mathcal{F}_1(x) \doteq \overline{co} \left\{ \sum_{\alpha=1}^m g_{\alpha}(x) w^{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha, \beta}(x) w^{\alpha} w^{\beta}; \quad (w^1, \dots, w^m) \in \mathbb{R}^m \right\}$$

depending on an additional parameter  $\xi \in \mathbb{R}^d$ , such that

$$f(\bar{x}) + \gamma(\bar{x}, \bar{\xi}) = 0. \quad (10.1)$$

for some  $\bar{\xi} \in \mathbb{R}^d$ . Assuming that  $\gamma$  is defined on an entire neighborhood of  $(\bar{x}, \bar{\xi})$ , consider the Jacobian matrices of partial derivatives computed at  $(\bar{x}, \bar{\xi})$ :

$$A \doteq \frac{\partial f}{\partial x} + \frac{\partial \gamma}{\partial x}, \quad B \doteq \frac{\partial \gamma}{\partial \xi}.$$

**Theorem 10.1** *In the above setting, if the linear system with constant coefficients*

$$\dot{x} = Ax + B\xi \quad (10.2)$$

*is completely controllable, then the differential inclusion (8.4)-(8.5) is weakly asymptotically stable at the point  $\bar{x}$ .*

We recall that the system (10.2) is completely controllable if and only if the matrices  $A, B$  satisfy the algebraic relation  $\text{Rank}[B, AB, \dots, A^{n-1}B] = n$ . This guarantees that the system can be steered from any initial state to any final state, within any given time interval [4, 32].

To prove the theorem, consider the control system

$$\dot{x} = f(x) + \gamma(x, \xi). \quad (10.3)$$

By a classical result in control theory, the above assumptions imply that, for every point  $x^{\#}$  sufficiently close to  $\bar{x}$ , there exists a trajectory starting from  $x^{\#}$  reaching  $\bar{x}$  in finite time. In particular, in view of (10.1), the system (10.3) is asymptotically stabilizable at the point  $\bar{x}$ . Since all trajectories of (10.3) are also trajectories of the differential inclusion (8.4), the result follows.  $\square$

**Remark 10.1** Toward the construction of smooth selections from the multifunction  $\mathcal{F}$  we observe that each closed convex set  $\mathcal{F}(x)$  can be equivalently written as

$$\begin{aligned} \mathcal{F}(x) &= f(x) + \overline{co} \left\{ \sum_{\alpha=1}^m g_{\alpha}(x) w^{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha, \beta}(x) w^{\alpha} w^{\beta}; \quad (w^1, \dots, w^m) \in \mathbb{R}^m \right\} \\ &\quad + \overline{co} \left\{ \sum_{\alpha, \beta=1}^m h_{\alpha, \beta}(x) w^{\alpha} w^{\beta}; \quad (w^1, \dots, w^m) \in \mathbb{R}^m \right\} \\ &\doteq f(x) + \mathcal{F}_1(x) + \mathcal{F}_2(x). \end{aligned} \quad (10.4)$$

Indeed, by definition we have  $\mathcal{F}(x) = f(x) + \mathcal{F}_1(x)$ . To establish the identity (10.4) it thus suffices to prove that

$$\mathcal{F}_1 + \mathcal{F}_2 \subseteq \mathcal{F}_1. \quad (10.5)$$



Since the set  $\mathcal{F}_1(x)$  is convex and contains the origin, for every  $(w^1, \dots, w^m) \in \mathbb{R}^m$  and  $\varepsilon \in [0, 1]$  we have

$$y_\varepsilon \doteq \varepsilon \left( \sum_{\alpha=1}^m g_\alpha(x) \frac{w^\alpha}{\sqrt{\varepsilon}} + \sum_{\alpha,\beta=1}^m h_{\alpha,\beta}(x) \frac{w^\alpha w^\beta}{\varepsilon} \right) \in \mathcal{F}_1.$$

Letting  $\varepsilon \rightarrow 0$  we find

$$\lim_{\varepsilon \rightarrow 0+} y_\varepsilon = \sum_{\alpha,\beta=1}^m h_{\alpha,\beta}(x) w^\alpha w^\beta. \quad (10.6)$$

Since  $\mathcal{F}_1(x)$  is closed, it must contain the right hand side of (10.6). This proves the inclusion  $\mathcal{F}_2 \subseteq \mathcal{F}_1$ . Next, observing that  $\mathcal{F}_2$  is a cone, for every  $y_2 \in \mathcal{F}_2$  and  $\varepsilon > 0$  we have  $\varepsilon^{-1}y_2 \in \mathcal{F}_2 \subseteq \mathcal{F}_1$ . Therefore, if  $y_1 \in \mathcal{F}_1$  we can write

$$y_1 + y_2 = \lim_{\varepsilon \rightarrow 0+} (1 - \varepsilon)y_1 + \varepsilon(\varepsilon^{-1}y_2) \in \mathcal{F}_1$$

because  $\mathcal{F}_1$  is closed and convex. This proves (10.5).

By Theorem 10.1 and the above remark, one may establish a stability result by constructing suitable selections  $\gamma(x, \xi) \in \mathcal{F}_2(x)$  from the cone  $\mathcal{F}_2$ .

## Part III

# Stabilization of mechanical systems

In this part we address the question of how to use some time-dependent holonomic constraints as controls in order to stabilize a mechanical system to a given state.

## 11 Stabilization with vibrating controls

For reader's convenience, we summarize the results in Section 4. Let  $G = (g_{r,s})_{r,s=1,\dots,N+M}$  be the matrix that represents the covariant inertial tensor in a given coordinate chart  $(q, u)$ . In particular, the kinetic energy of the whole system at a state  $(q, u)$  with velocity  $(v, w) \in \mathbb{R}^{N+M}$  is given by

$$\mathcal{T} = \frac{1}{2}g_{i,j}(q, u)v^i v^j + g_{i,N+\alpha}(q, u)v^i w^\alpha + \frac{1}{2}g_{N+\alpha,N+\beta}(q, u)w^\alpha w^\beta.$$

Here and in the sequel,  $i, j = 1, \dots, N$  while  $\alpha, \beta = 1, \dots, M$ . By  $G^{-1} = (g^{r,s})_{r,s=1,\dots,N+M}$  we denote the inverse of  $G$ . Moreover, we consider the sub-matrices  $G_1 \doteq (g_{i,j})$ ,  $(G^{-1})_2 \doteq (g^{N+\alpha,N+\beta})$ , and  $(G^{-1})_{12} \doteq (g^{i,N+\alpha})$ . Finally, we introduce the matrices

$$A = (a^{i,j}) \doteq (G_1)^{-1}, \quad E = (e_{\alpha,\beta}) \doteq ((G^{-1})_2)^{-1}, \quad K = (k_\alpha^i) \doteq (G^{-1})_{12}E. \quad (11.1)$$

We recall that all the above matrices depend on the variables  $q, u$ . Concerning the external force, our main assumption will be

**Hypothesis (A).** *The force  $F$  acting on the whole system does not explicitly depend on time, and is affine w.r.t. the time derivative of the control, so that*

$$F = F(q, p, u, w) = F^0(q, p, u) + F^1(q, p, u) \cdot w. \quad (11.2)$$

Taking the component along the manifold  $\mathcal{Q}$ , this implies

$$F_{\mathcal{Q}} = F_{\mathcal{Q}}(q, p, u, w) = F_{\mathcal{Q}}^0(q, p, u) + F_{\mathcal{Q}}^1(q, p, u) \cdot w.$$

We can thus write the equations of motion in the form

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} Ap \\ -\frac{1}{2}p^\dagger \frac{\partial A}{\partial q} p + F_{\mathcal{Q}}^0 \\ 0 \end{pmatrix} + \begin{pmatrix} K \\ -p^\dagger \frac{\partial K}{\partial q} + F_{\mathcal{Q}}^1 \\ 1_M \end{pmatrix} \dot{u} + \dot{u}^\dagger \begin{pmatrix} 0 \\ \frac{1}{2} \frac{\partial E}{\partial q} \\ 0 \end{pmatrix} \dot{u}. \quad (11.3)$$

Our main goal is to find conditions which imply that the system (11.3) is stabilizable at a point  $(\bar{q}, 0, \bar{u})$ . Two results will be described here. The first one relies on suitable smooth selections from the corresponding set-valued maps, as in Theorem 10.1. The second one is based on the use of Lyapunov functions.

For each  $q, u$ , consider the cone

$$\Gamma(q, u) \doteq \overline{\text{co}} \left\{ w^\dagger \frac{\partial E(q, u)}{\partial q} w ; \quad w \in \mathbb{R}^M \right\}. \quad (11.4)$$

Let  $\xi \in \mathbb{R}^d$  be an auxiliary control variable, ranging on a neighborhood of a point  $\bar{\xi} \in \mathbb{R}^d$ , and consider a control system of the form

$$\begin{cases} \dot{q} = Ap, \\ \dot{p} = F_{\mathcal{Q}}^0(q, p, \bar{u}) + \gamma(q, p, \bar{u}, \xi), \end{cases} \quad (11.5)$$

where  $\gamma$  is a suitable selection from the cone  $\Gamma$ . It will be convenient to write (11.5) in the more compact form

$$(\dot{q}, \dot{p}) = \Phi(q, p, \bar{u}, \xi), \quad (11.6)$$

regarding  $(q, p) \in \mathbb{R}^{N+N}$  as state variables and  $\xi \in \mathbb{R}^d$  as control variable. Assume that

$$F_{\mathcal{Q}}^0(\bar{q}, 0, \bar{u}) + \gamma(\bar{q}, 0, \bar{u}, \bar{\xi}) = 0. \quad (11.7)$$

By (11.5) this implies  $\Phi(\bar{q}, 0, \bar{u}, \bar{\xi}) = 0 \in \mathbb{R}^{2N}$ . To test the local controllability of (11.5) at the equilibrium point  $(\bar{q}, 0, \bar{u}, \bar{\xi})$  we look at the linearized system with constant coefficients

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \Lambda \begin{pmatrix} p \\ q \end{pmatrix} + \mathcal{B}\xi, \quad (11.8)$$

where

$$\Lambda = \frac{\partial \Phi}{\partial (q, p)} \quad \mathcal{B} = \frac{\partial \Phi}{\partial \xi}$$

with all partial derivatives being computed at the point  $(\bar{q}, 0, \bar{u}, \bar{\xi})$ . We can now state

**Theorem 11.1** *Assume that a smooth map*

$$(q, p, u, \xi) \mapsto \gamma(q, p, u, \xi) \in \Gamma(q, u) \quad (11.9)$$

*can be chosen in such a way that (11.7) holds and so that the linear system (11.8) is completely controllable. Then the system (11.3) is asymptotically stabilizable at the point  $(\bar{q}, 0, \bar{u})$ .*

**Proof.** According to Theorem 10.1 and Remark 10.1, it suffices to show that the control system

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} Ap \\ -\frac{1}{2}p^\dagger \frac{\partial A}{\partial q} p + F_Q^0 \\ 0 \end{pmatrix} + \begin{pmatrix} K \\ -p^\dagger \frac{\partial K}{\partial q} + F_Q^1 \\ 1_M \end{pmatrix} w + w^\dagger \begin{pmatrix} 0 \\ \frac{1}{2} \frac{\partial E}{\partial q} \\ 0 \end{pmatrix} w + \begin{pmatrix} 0 \\ \gamma(q, p, u, \xi) \\ 0 \end{pmatrix} \quad (11.10)$$

is locally controllable at  $(\bar{q}, 0, \bar{u})$ . Notice that in (11.10) the state variables are  $q, p, u$ , while  $w, \xi$  are the controls. Computing the Jacobian matrices of partial derivatives at the point  $(q, p, u; w, \xi) = (\bar{q}, 0, \bar{u}, 0, \bar{\xi})$ , we obtain a linear system with constant coefficients, of the form

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & 0 & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \\ u \end{pmatrix} + \begin{pmatrix} 0 & B_{12} \\ B_{21} & B_{22} \\ 0 & 1_M \end{pmatrix} \begin{pmatrix} \xi \\ w \end{pmatrix} \doteq \tilde{\Lambda} \begin{pmatrix} q \\ p \\ u \end{pmatrix} + (\tilde{\mathcal{B}}_1 \quad \tilde{\mathcal{B}}_2) \begin{pmatrix} \xi \\ w \end{pmatrix} \quad (11.11)$$

By assumption, the linear system (11.8) is completely controllable. Therefore

$$\text{Rank} \left[ \mathcal{B}, \Lambda \mathcal{B}, \dots, \Lambda^{2N-1} \mathcal{B} \right] = 2N. \quad (11.12)$$

We now observe that the matrices  $\Lambda, \mathcal{B}$  at (11.8) correspond to the submatrices

$$\Lambda = \begin{pmatrix} \Lambda_{11} & 0 \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 \\ B_{21} \end{pmatrix}. \quad (11.13)$$

Hence from (11.12) it follows

$$\text{span} \left[ \tilde{\mathcal{B}}_1, \tilde{\Lambda} \tilde{\mathcal{B}}_1, \dots, \tilde{\Lambda}^{2N-1} \tilde{\mathcal{B}}_1 \right] = \left\{ \begin{pmatrix} q \\ p \\ 0 \end{pmatrix}; \quad q \in \mathbb{R}^N, p \in \mathbb{R}^N \right\}. \quad (11.14)$$

Adding to this subspace the subspace generated by the columns of the matrix  $\tilde{\mathcal{B}}_2$ , we obtain the entire space  $\mathbb{R}^{2N+M}$ . We thus conclude that the linear system (11.11) is completely controllable. In turn, this implies that the nonlinear system (11.10) is asymptotically stabilizable at  $(\bar{q}, 0, \bar{u})$ , completing the proof.  $\square$

By choosing a special kind of selection and relying of the particular structure of (11.5), we can deduce Corollary 11.1 below. To state it, if  $k$  is a positive integer such that  $kM \geq N$  and

$W = (w_1, \dots, w_k) \in \mathbb{R}^{M \times k}$ , let us consider the  $N \times kM$  matrix

$$M(u, q, W) \doteq \begin{pmatrix} \frac{\partial e_{1,\beta}}{\partial q^1} w_1^\beta, \dots, \frac{\partial e_{M,\beta}}{\partial q^1} w_1^\beta, & \dots, & \frac{\partial e_{1,\beta}}{\partial q^1} w_k^\beta, \dots, \frac{\partial e_{M,\beta}}{\partial q^1} w_k^\beta \\ & \dots & \\ \frac{\partial e_{1,\beta}}{\partial q^N} w_1^\beta, \dots, \frac{\partial e_{M,\beta}}{\partial q^N} w_1^\beta, & \dots, & \frac{\partial e_{1,\beta}}{\partial q^N} w_k^\beta, \dots, \frac{\partial e_{M,\beta}}{\partial q^N} w_k^\beta \end{pmatrix}. \quad (11.15)$$

**Corollary 11.1** *Let  $k$  be a positive integer and assume that for a given state  $(\bar{q}, \bar{u})$  there exists a  $k$ -tuple  $\bar{W} = (\bar{w}_1, \dots, \bar{w}_k) \in (R^M)^k$  such that*

$$\text{Rank}(M(\bar{u}, \bar{q}, \bar{W})) = N \quad (11.16)$$

and

$$\begin{cases} (F_Y^0)^1 + \sum_{\alpha,\beta=1}^M \frac{\partial e_{\alpha,\beta}}{\partial q^1} \sum_{r=1}^k \bar{w}_r^\alpha \bar{w}_r^\beta = 0 \\ \dots \\ (F_Y^0)^N + \sum_{\alpha,\beta=1}^M \frac{\partial e_{\alpha,\beta}}{\partial q^N} \sum_{r=1}^k \bar{w}_r^\alpha \bar{w}_r^\beta = 0, \end{cases} \quad (11.17)$$

where the involved functions are computed at  $(q, p, u) = (\bar{q}, 0, \bar{u})$ . Then the system (11.3) is asymptotically stabilizable at the point  $(\bar{q}, 0, \bar{u})$ .

**Proof.** Let us observe that the matrices  $\Lambda$  and  $\mathcal{B}$  in (11.13) have the following form:

$$\mathcal{B} = \begin{pmatrix} 0_{N \times d} \\ \frac{\partial \gamma}{\partial \xi} \end{pmatrix} \quad \Lambda = \begin{pmatrix} 0_{N \times N} & A \\ \frac{\partial(F+\gamma)}{\partial q} & \frac{\partial(F+\gamma)}{\partial p} \end{pmatrix} \quad (11.18)$$

so that, in particular,

$$\Lambda \mathcal{B} = \begin{pmatrix} A \cdot \frac{\partial \gamma}{\partial \xi} \\ \frac{\partial(F+\gamma)}{\partial p} \cdot \frac{\partial \gamma}{\partial \xi} \end{pmatrix} \quad (11.19)$$

Let us set  $d = kM$ ,  $\xi = W = (w_1, \dots, w_k)$ , and

$$\gamma_i(q, u, W) \doteq \frac{1}{2} \sum_{\ell=1}^k \frac{\partial e_{\alpha,\beta}}{\partial q^i} w_\ell^\alpha w_\ell^\beta \quad i = 1, \dots, N$$

Notice that, by 2-homogeneity  $\gamma = (\gamma^1, \dots, \gamma^N)$ , is in fact a selection of the set-valued map  $\Gamma$  defined in (11.4). In view of Theorem 11.1, to prove the asymptotic stability it is sufficient find

$$\bar{\xi} = \bar{W}$$

such (11.17) holds and, moreover,

$$\text{Rank} [\mathcal{B}, \Lambda \mathcal{B}] (\bar{q}, 0, \bar{u}, \bar{W}) = 2N.$$

Since  $A$  is a non-singular matrix, by (11.19) the latter condition is equivalent to

$$\text{Rank} \left( \frac{\partial \gamma}{\partial W} \right) (\bar{q}, 0, \bar{u}, \bar{W}) = N. \quad (11.20)$$

In turn, this coincides with (11.16), so the proof is concluded.  $\square$

We now describe a second approach, based on the construction of a Lyapunov function. Throughout the following we assume that the external force  $F$  in (11.2) admits the representation

$$F = F(q, p, u, w) = -\frac{\partial U}{\partial(q, u)} + F^1(q, p, u) \cdot w. \quad (11.21)$$

in terms of a potential function  $U = U(q, u)$ .

**Definition 11.1** *Given a  $k$ -tuple of vectors  $W \doteq \{w_1, \dots, w_k\} \subset \mathbb{R}^M$ , the corresponding asymptotic effective potential  $(q, u) \mapsto U_W(q, u)$  is defined as*

$$U_W(q, u) \doteq U(q, u) - \frac{1}{2} \sum_{\ell=1}^k w_\ell^\dagger E(q, u) w_\ell$$

$$\left( = U(q, u) - \frac{1}{2} \sum_{\ell=1}^k \sum_{\alpha, \beta=1}^M e_{\alpha, \beta}(q, u) w_\ell^\alpha w_\ell^\beta \right).$$

**Theorem 11.2** . *Let the external force  $F$  have the form (11.21). For a given state  $(\bar{q}, \bar{u})$ , assume that there exist a neighborhood  $\mathcal{N}$  of  $(\bar{q}, \bar{u})$  and a  $k$ -tuple  $W \doteq \{w_1, \dots, w_k\} \subset \mathbb{R}^M$ , as in Definition 11.1 which, in addition, satisfy the following property:*

- *There exists a continuously differentiable map  $u \mapsto \beta(u)$  defined on a neighborhood of  $\bar{u}$  such that the function*

$$(q, u) \mapsto U_W(q, u) + \beta(u)$$

*has a strict local minimum at  $(q, u) = (\bar{q}, \bar{u})$ .*

*Then the system (11.3) is stabilizable at  $(\bar{q}, 0, \bar{u})$ .*

**Proof.** As in Section 10, consider the symmetrized differential inclusion corresponding to (11.3), namely

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{z} \end{pmatrix} \in \overline{co} \left\{ \begin{pmatrix} Ap \\ -\frac{1}{2} p^\dagger \frac{\partial A}{\partial q} p - \frac{\partial U}{\partial q} \\ 0 \end{pmatrix} + w^\dagger \begin{pmatrix} 0 \\ \frac{1}{2} \frac{\partial E}{\partial q} \\ 0 \end{pmatrix} w, \quad w \in \mathbb{R}^M \right\}. \quad (11.22)$$

To prove the theorem, it suffices to show that the point  $(\bar{q}, 0, \bar{u})$  is a stable equilibrium for the differential equation

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} Ap \\ -\frac{1}{2} p^\dagger \frac{\partial A}{\partial q} p - \frac{\partial U_W}{\partial q} \\ 0 \end{pmatrix}. \quad (11.23)$$

Indeed, by the definition of  $U_W$ , the right hand side of (11.23) is a selection of the right hand-side of (11.22). Introducing the Hamiltonian function

$$H_W \doteq \frac{1}{2} p A p^\dagger + U_W,$$

the equation (11.23) can be written in the following Hamiltonian form:

$$\begin{pmatrix} \dot{q} & \dot{p} & \dot{u} \end{pmatrix}^\dagger = \begin{pmatrix} \frac{\partial H_W}{\partial p} & -\frac{\partial H_W}{\partial q} & 0 \end{pmatrix}. \quad (11.24)$$

Therefore the map

$$V(q, p, u) \doteq H_W(q, p, u) + \beta(u) \quad (11.25)$$

is a Lyapunov function for (11.23), from which it follows that  $(\bar{q}, 0, \bar{z})$  is a stable equilibrium for (11.23).  $\square$

## 12 Examples

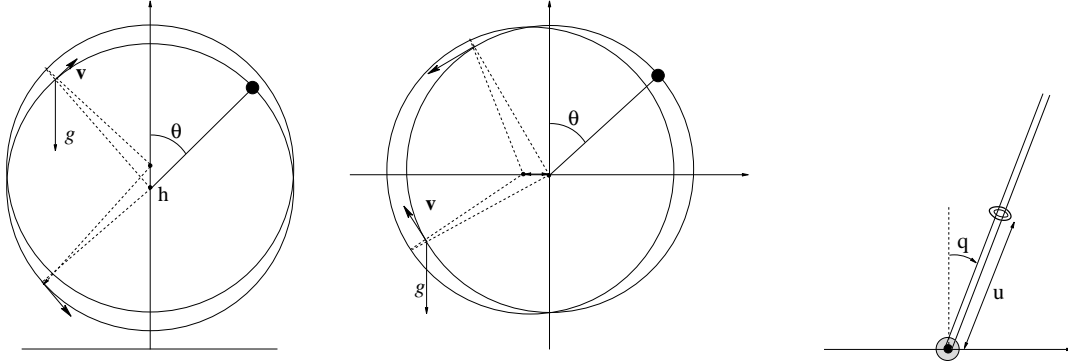


Figure 3: A pendulum whose pivot oscillates vertically (on the left) and horizontally (center). On the right: a bead sliding without friction along a rotating axis.

**Example 1 (pendulum with oscillating pivot).** Let us consider a pendulum with fixed length  $r = 1$ , whose pivot is moving on the vertical  $y$ -axis, as shown in Figure 3, left. Its position is described by two variables: the clockwise angle  $\theta$  formed by the pendulum with the  $y$ -axis, and the height  $h$  of the pivot. We now consider  $h = u(t)$  to be our control variable, while the evolution of the other variable  $\theta = q(t)$  will be determined by the equations of motion. We assume that the control function  $t \mapsto u(t)$  can be assigned as a function of time, ranging over a neighborhood of the origin. We assume that the both the pendulum and its pivot have unit mass, so that the kinetic matrix  $G$  and the matrices in (11.1) take the form

$$G = \begin{pmatrix} 1 & -\sin q \\ -\sin q & 2 \end{pmatrix} \quad A = (1), \quad E = (1 + \cos^2 q), \quad K = (\sin q).$$

**Remark 12.1** To be consistent with the general theory we need to put a mass on the pivot as well. This is needed in order that the matrix  $G$  be invertible. On the other hand it is easy to show that the resulting control equations are independent of the mass of the pivot. Actually this should be expected, since the motion of the pivot is here considered as a control. Of course, what is not independent of the mass of the pivot is the constraint reaction necessary to produce a given motion of  $u$ .

Notice that orthogonal curvature of the constraint foliation  $\Lambda$  —i.e. the coefficient of  $(\dot{u})^2$ , see Section 6— is different from zero, for  $\frac{dE}{dq} = -2 \sin q \cos q$ .

In the presence of downward gravitational acceleration  $g$ , the control equations for  $q$  and the

corresponding momentum  $p$  is given by

$$\begin{cases} \dot{q} = p + (\sin q)\dot{u} \\ \dot{p} = -\frac{\partial U}{\partial q} - p(\cos q)\dot{u} - (\sin q)(\cos q)\dot{u}^2, \end{cases} \quad (12.1)$$

where  $U(q, u) \doteq g \cos q$  is the gravitational potential.

Using Theorem 11.2, it is easy to check that this system is stabilizable at the upward equilibrium point  $(\bar{q}, \bar{p}, \bar{u}) = (0, 0, 0)$ . Indeed, choosing  $W = \{w\}$  with  $w > g$ , the corresponding effective potential

$$U_W = g \cos q - \frac{1}{2}(1 + \cos^2 q)w^2.$$

has a strict local minimum at  $q = 0$ .

To illustrate an application of Theorem 11.1, we now show that the above system is asymptotically stabilizable at every position  $(\bar{q}, 0, 0)$  with  $0 < |\bar{q}| < \pi/2$ . To fix the ideas, assume  $\bar{q} > 0$ , the other case being entirely similar. For  $\xi > 0$ , the map  $\gamma(q, p, \xi) = -\xi$  provides a smooth selection from the cone

$$\Gamma(q, u) \doteq \overline{co} \left\{ \frac{\partial E(q, u)}{\partial q} w^2; \quad w \in \mathbb{R} \right\} = \{-\xi; \quad \xi \geq 0\}.$$

The corresponding system (11.5), with  $\xi$  as control variable, now takes the form

$$\begin{cases} \dot{q} = p \\ \dot{p} = g \sin q - \xi. \end{cases} \quad (12.2)$$

It is easy to check that  $(\bar{q}, \bar{p}, \bar{\xi}) = (\bar{q}, 0, g \sin \bar{q})$  is an equilibrium position and the system is locally controllable at this point. Indeed, the linearized control system with constant coefficients is

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g \cos \bar{q} & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \xi.$$

By Theorem 11.1, the system (12.1) is asymptotically stabilizable at  $(\bar{q}, 0, 0)$ .

By similar arguments one can show that, by means of horizontal oscillations of the pivot, one can stabilize the system at any position of the form  $(\bar{q}, 0, 0)$ , with  $\frac{\pi}{2} \leq |\bar{q}| \leq \pi$ .

**Example 2 (sliding bead).** Consider the mechanical system represented in Figure 3 (right), consisting of a bead sliding without friction along a bar, and subject to gravity. The bar can be rotated around the origin, in a vertical plane. Calling  $q$  the distance of the bead from the origin, while  $u$  is the angle formed by the bar with the vertical line. Regarding  $u$  as the controlled variable, in this case the kinetic matrix  $G$  and the matrices in (11.1) take the form

$$G = \begin{pmatrix} 1 & 0 \\ 0 & q^2 \end{pmatrix}, \quad A = (1), \quad E = (q^2), \quad K = (0).$$

The orthogonal curvature of the constraint foliation  $\Lambda$  is not vanishing identically: indeed, one has  $\frac{dE}{dq} = 2q$ . The control equations for  $q$  and the corresponding momentum  $p$  are

$$\begin{cases} \dot{q} = p, \\ \dot{p} = -g \cos u + q\dot{u}^2. \end{cases} \quad (12.3)$$

This case is more intuitive than the previous ones. Indeed, it is clear that a rapid oscillation of the angle  $u$  generates a centrifugal force that can contrast the gravitational force. More precisely, the system can be asymptotically stabilized at each  $(\bar{q}, \bar{p}, \bar{u}) \in ]0, +\infty[ \times \{0\} \times ]-\pi/2, \pi/2[$ . A simple proof of this fact follows from Theorem 11.1. Indeed, for  $q > 0$  we trivially have  $\Gamma(q, u) = \{qw^2; w \in \mathbb{R}\} = \{\xi \in \mathbb{R}; \xi \geq 0\}$ . It is now clear that, if  $\cos \bar{u} > 0$ , then the control system

$$\begin{cases} \dot{q} = p, \\ \dot{p} = -g \cos \bar{u} + \xi, \end{cases}$$

admits the equilibrium point  $(\bar{q}, 0, \bar{\xi})$ , with  $\bar{\xi} = g \cos \bar{u} > 0$ . Moreover, this system is completely controllable around this equilibrium point, using with controls  $\xi \geq 0$ . An application of Theorem 11.1 yields the asymptotic stability property.

We remark that the stabilizing controls cannot be independent of the position  $q$  and the velocity  $p$ . In particular, the approach in Theorem 11.2, based on effective potential, cannot be pursued in this case, because a constant control  $w$  cannot stabilize the system

$$\begin{cases} \dot{q} = p, \\ \dot{p} = -g \cos u + qw^2. \end{cases}$$

**Example 3 (double pendulum with moving pivot).** So far we have considered examples with scalar controls. We wish now to study a case where the control  $u$  is two-dimensional, hence the cone (11.4) is also two-dimensional. Consider a double pendulum consisting of three point masses  $P_0, P_1, P_2$ , such that the distances  $|P_0P_1|, |P_1P_2|$  are fixed, say both equal to 1. Let these points be subject to the gravitational force and constrained without friction on a vertical plane. Let  $(u^1, u^2)$  be the cartesian coordinates of the pivot  $P_0$ , and let  $q^1, q^2$  the clockwise angles formed by  $P_0P_1$  and  $P_1P_2$  with the upper vertical half lines centered in  $P^0$  and  $P^1$ , respectively, see Figure 4. Because of the constraints, the state of the system  $\{P_0, P_1, P_2\}$  is thus entirely described by the four coordinates  $(q^1, q^2, u^1, u^2)$ . The reduced system, obtained by regarding the parameters  $(u^1, u^2)$  as controls and the coordinates  $(q^1, q^2)$  as state-coordinates, is two-dimensional. We assume that the all three points have unit mass, so that the matrix  $G = (g_{rs})$  representing the kinetic energy is given by

$$G = \begin{pmatrix} 2 & \cos(q^1 - q^2) & 2 \cos q^1 & -2 \sin q^1 \\ \cos(q^1 - q^2) & 1 & \cos q^2 & -\sin q^2 \\ 2 \cos q^1 & \cos q^2 & 3 & 0 \\ -2 \sin q^1 & -\sin q^2 & 0 & 3 \end{pmatrix},$$

Moreover, recalling (11.1), we have

$$E = \begin{pmatrix} 1 - \frac{4(\sin q^1)^2}{-3 + \cos 3(q^1 - q^2)} & -\frac{2 \sin 2q^1}{-3 + \cos 3(q^1 - q^2)} \\ -\frac{2 \sin 2q^1}{-3 + \cos 3(q^1 - q^2)} & 1 - \frac{4(\sin q^1)^2}{-3 + \cos 3(q^1 - q^2)} \end{pmatrix},$$

$$(F_Q^0)^1 = 2g \sin q^1, \quad (F_Q^0)^2 = g \sin q^2.$$



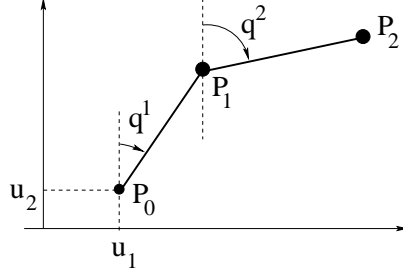


Figure 4: Controlling the double pendulum by moving the pivot at  $P_0$ .

Let us observe, as in Remark 12.1, that the matrix  $E$  and the corresponding control equations are independent of the pivot's mass.

**Proposition 12.1** *For every  $\bar{q}^1 \in ]0, \pi/4[$  (resp.  $\bar{q}^1 \in ]-\pi/4, 0[$ ) there exists  $\delta > 0$  such that for all  $\bar{q}^2 \in ]-\delta, 0[$  (resp.  $\bar{q}^2 \in ]-\delta, 0[$ ) the system is stabilizable at  $(q^1, q^2, p^1, p^2, u^1, u^2) = (\bar{q}^1, \bar{q}^2, 0, 0, 0, 0)$ .*

*Moreover, the system is stabilizable at  $(q^1, q^2, p^1, p^2, u^1, u^2) = (0, 0, 0, 0, 0, 0)$ .*

**Remark 12.2** For obvious reasons of translational invariance, if we replace  $(u_1, u_2) = (0, 0)$  with any other value  $(\bar{u}^1, \bar{u}^2) \in \mathbb{R}^2$  the result holds true as well.

**Proof of Proposition 12.1.** Using Corollary 11.1 with  $N = M = 2$  and  $k = 1$ , we have that the system can be stabilized to  $(\bar{q}^1, \bar{q}^2, \bar{u}^1, \bar{u}^2)$  provided there exist  $\bar{w} \in \mathbb{R}^2$  such that

$$\begin{cases} 2g \sin \bar{q}^1 + \sum_{\alpha, \beta=1}^2 \frac{\partial e_{\alpha, \beta}}{\partial \bar{q}^1} \bar{w}^\alpha \bar{w}^\beta = 0 \\ g \sin \bar{q}^2 + \sum_{\alpha, \beta=1}^2 \frac{\partial e_{\alpha, \beta}}{\partial \bar{q}^2} \bar{w}^\alpha \bar{w}^\beta = 0 \end{cases} \quad (12.4)$$

and

$$\det \begin{pmatrix} \frac{\partial e_{1,1}}{\partial \bar{q}^1} \bar{w}^1 + \frac{\partial e_{1,2}}{\partial \bar{q}^1} \bar{w}^2 & \frac{\partial e_{2,1}}{\partial \bar{q}^1} \bar{w}^1 + \frac{\partial e_{2,2}}{\partial \bar{q}^1} \bar{w}^2 \\ \frac{\partial e_{1,1}}{\partial \bar{q}^2} \bar{w}^1 + \frac{\partial e_{1,2}}{\partial \bar{q}^2} \bar{w}^2 & \frac{\partial e_{2,1}}{\partial \bar{q}^2} \bar{w}^1 + \frac{\partial e_{2,2}}{\partial \bar{q}^2} \bar{w}^2 \end{pmatrix} \neq 0 \quad (12.5)$$

Notice that the latter relation can be written as

$$Q_{\alpha, \beta} \bar{w}^\alpha \bar{w}^\beta \neq 0 \quad (12.6)$$

where the matrix  $Q = (Q_{\alpha, \beta})$  is defined by

$$Q \doteq \frac{\partial E}{\partial \bar{q}^1} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \frac{\partial E}{\partial \bar{q}^2}. \quad (12.7)$$

We recall that  $E$  denotes the matrix  $(e_{\alpha, \beta})$ . Moreover, it is meant that the functions in (12.4)-(12.7) are computed at  $(\bar{q}^1, \bar{q}^2)$ .

Let us fix  $\bar{q}^1 \in ]0, \pi/4[$ . In order to establish the existence of a  $\delta > 0$  such that for every  $\bar{q}^2 \in ]-\delta, 0[$  there is a  $\bar{w}$  verifying the relations (12.4), (12.5), we need to study the intersections of the level sets of the quadratic forms  $Q, \frac{\partial E}{\partial \bar{q}^1}, \frac{\partial E}{\partial \bar{q}^2}$ .

Let us write the matrix  $\frac{\partial E}{\partial q^1}$  and  $\frac{\partial E}{\partial q^2}$  explicitly:

$$\frac{\partial E}{\partial q^1} = \begin{pmatrix} \frac{8 \sin q^1 (-3 \cos q^1 + \cos(q^1 - 2q^2))}{(-3 + \cos(2(q^1 - q^2)))^2} & -\frac{4(-3 \cos 2q^1 + \cos 2q^2)}{(-3 + \cos(2(q^1 - q^2)))^2} \\ -\frac{4(-3 \cos 2q^1 + \cos 2q^2)}{(-3 + \cos(2(q^1 - q^2)))^2} & -\frac{8 \cos q^1 (3 \sin q^1 + \sin(q^1 - 2q^2))}{(-3 + \cos(2(q^1 - q^2)))^2} \end{pmatrix}$$

$$\frac{\partial E}{\partial q^2} = \begin{pmatrix} \frac{8 \sin^2 q^1 \sin(2(q^1 - q^2))}{(-3 + \cos(2(q^1 - q^2)))^2} & \frac{4 \sin 2q^1 \sin(2(q^1 - q^2))}{(-3 + \cos(2(q^1 - q^2)))^2} \\ \frac{4 \sin 2q^1 \sin(2(q^1 - q^2))}{(-3 + \cos(2(q^1 - q^2)))^2} & \frac{8 \cos^2 q^1 \sin(2(q^1 - q^2))}{(-3 + \cos(2(q^1 - q^2)))^2} \end{pmatrix}$$

In particular, one has

$$\det \left( \frac{\partial E}{\partial q^1}(q^1, q^2) \right) = -\frac{16}{(-3 + \cos(2(q^1 - q^2)))^2} < 0, \quad \det \left( \frac{\partial E}{\partial q^2}(q^1, q^2) \right) = 0$$

for all  $q^1, q^2$ .

Hence, one has:

- (i) The quadratic form  $w \mapsto w^\dagger \frac{\partial E}{\partial q^1} w$  is indefinite, so it can be factorized by two linear, independent, forms. Let us assume that,  $\bar{q}^2 \in ]-\bar{q}^1, 0[$ , so that, in particular,  $\frac{\partial e_{2,2}}{\partial q^1} < 0$ . Hence, for suitable functions  $a = a(q^1, q^2)$ ,  $b = b(q^1, q^2)$  such that  $a(q^1, q^2) \neq b(q^1, q^2)$  for all  $q^1, q^2$ , one has

$$\frac{\partial e_{\alpha,\beta}}{\partial q^1} w^\alpha w^\beta = \frac{\partial e_{2,2}}{\partial q^1} (w^2 - aw^1)(w^2 - bw^1).$$

- (ii) If  $\bar{q}^2 \in ]-\bar{q}^1, 0[$ , the quadratic form  $w \mapsto w^\dagger \frac{\partial E}{\partial q^2} w$  is positive semi-definite. Hence it can be factorized by the positive scalar function  $\frac{\partial e_{2,2}}{\partial q^1}$  and the square of a linear function. Moreover *this linear function coincides with one of the two linear factors of the quadratic form  $w \mapsto w^\dagger \frac{\partial E}{\partial q^1} w$* . This is a trivial consequence of the identity

$$\left( \frac{\partial e_{1,2}}{\partial q^2} \frac{\partial e_{2,2}}{\partial q^1} \right)^2 - 2 \frac{\partial e_{2,1}}{\partial q^1} \frac{\partial e_{2,2}}{\partial q^1} \frac{\partial e_{1,2}}{\partial q^2} \frac{\partial e_{2,2}}{\partial q^1} \frac{\partial e_{2,2}}{\partial q^2} + \frac{\partial e_{1,1}}{\partial q^1} \frac{\partial e_{2,2}}{\partial q^1} \left( \frac{\partial e_{2,2}}{\partial q^2} \right)^2 = 0,$$

which can be verified by direct computation. Let  $(w^2 - aw^1)$  be the common factor of the two quadratic forms. Hence, we obtain

$$\frac{\partial e_{\alpha,\beta}}{\partial q^2} w^\alpha w^\beta = \frac{\partial e_{2,2}}{\partial q^2} (w^2 - aw^1)^2.$$

- (iii) The quadratic form  $w \mapsto w^\dagger Q w$  is semi-definite and, at each  $(q^1, q^2)$ , *it is proportional to the form  $w^\dagger \frac{\partial E}{\partial q^2} w$* . More precisely, one has

$$Q_{\alpha,\beta} w^\alpha w^\beta = \left( \frac{\partial e_{2,2}}{\partial q^1} \cdot \frac{a-b}{2} \right) \frac{\partial e_{\alpha,\beta}}{\partial q^2} w^\alpha w^\beta = \left( \frac{\partial e_{2,2}}{\partial q^1} \cdot \frac{\partial e_{2,2}}{\partial q^2} \cdot \frac{a-b}{2} \right) (w^2 - aw^1)^2.$$

This is easily deduced by (12.7). Notice, in particular, the form  $Q_{\alpha,\beta}w^\alpha w^\beta$  is never equal to the null form, since  $a(q^1, q^2) \neq b(q^1, q^2)$  for all  $q^1, q^2$ .

If  $S$  is a  $2 \times 2$  matrix and  $\rho \in \mathbb{R}$  let us set

$$\{w^\dagger S w = \rho\} \doteq \{w \in \mathbb{R}^2 \mid w^\dagger S w = \rho\}.$$

Since  $w^\dagger \frac{\partial E}{\partial q^2} w$  is positive definite and  $\sin q^2 < 0$ , there exists a real number  $\eta > 0$  such that

$$\left\{w^\dagger \frac{\partial E}{\partial q^2} w = -\sin \bar{q}^2\right\} = \{w \in \mathbb{R}^2 : (w^2 - aw^1) = \eta\} \cup \{w \in \mathbb{R}^2 : (w^2 - aw^1) = -\eta\},$$

so that, in particular,

$$\left\{w^\dagger \frac{\partial E}{\partial q^2} w = -g \sin \bar{q}^2\right\} \cap \{w \in \mathbb{R}^2 : (w^2 - aw^1) = 0\} = \emptyset.$$

By (iii) this implies

$$\left\{w^\dagger \frac{\partial E}{\partial q^2} w = -g \sin \bar{q}^2\right\} \cap \{w^\dagger Q w = 0\} = \emptyset. \quad (12.8)$$

Moreover, by (i) the line  $\{w \in \mathbb{R}^2 : (w^2 - aw^1) = 0\}$  is asymptotic to the hyperbolic arc

$$\left\{w^\dagger \frac{\partial E}{\partial q^1} w = -2g \sin \bar{q}^1\right\},$$

which implies

$$\left\{w^\dagger \frac{\partial E}{\partial q^1} w = -2g \sin \bar{q}^1\right\} \cap \left\{w^\dagger \frac{\partial E}{\partial q^2} w = -g \sin \bar{q}^2\right\} \neq \emptyset. \quad (12.9)$$

Putting (12.8) and (12.9) together, we obtain the first statement of the theorem.

On the other hand, the second statement will be proved by an application of Theorem 11.2. Since  $U(q) = g(2 \cos q^1 + \cos q^2)$  is a potential, by letting  $W = \{(0, \eta)\}$  and  $\beta(u) \doteq (u^1)^2 + (u^2)^2$ , we have that the effective potential

$$U_W(q, u) \doteq U(q) + \eta^2 e_{2,2}(q) + \beta(u)$$

has a strict minimum at  $(q, u) = (0, 0, 0, 0)$  as soon as  $|\eta|$  is large enough. In view of Theorem 11.2, this implies that the system is stabilizable at  $(q^1, q^2, p_1, p_2, u^1, u^2) = (0, 0, 0, 0, 0, 0)$ .

□

## Acknowledgements

The work of the first author was supported by the N.S.F., Grant DMS-0505430.

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